Introduction to Multicopter Design and Control

Lesson 05 Coordinate System and Attitude Representation

Quan Quan , Associate Professor

qq_buaa@buaa.edu.cn

BUAA Reliable Flight Control Group, http://rfly.buaa.edu.cn/

Beihang University, China

北航可靠飞行控制研究组
BUAA Reliable Flight Control Group

Preface

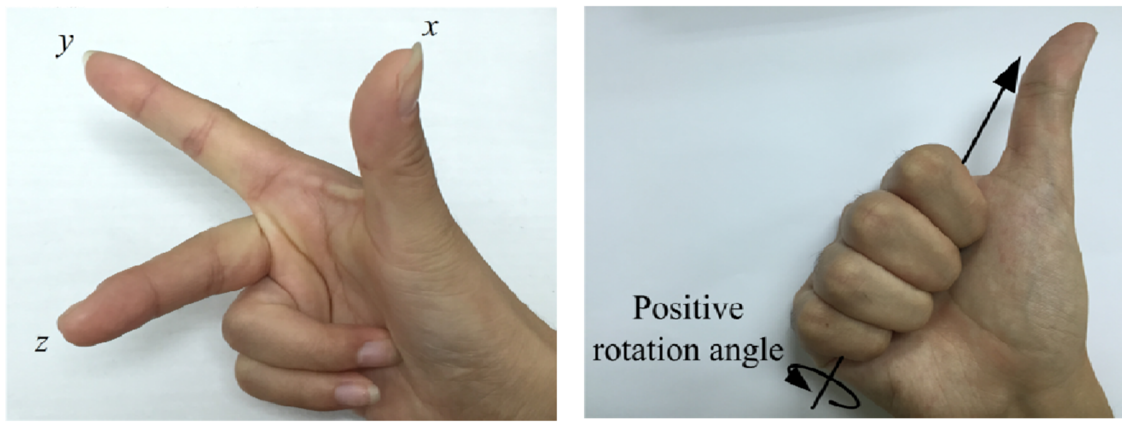
What are the three attitude representation methods and the relationship between their derivatives and the aircraft body's angular velocity?

Outline

1. Coordinate System
2. Attitude Representation
   – Euler Angles
   – Rotation matrix
   – Quaternion
3. Conclusion

1. Coordinate System

□ Right-Hand Rule



(a) Coordinate axes (b) Positive direction of a rotation

Fig 5.1 Coordinate axes and the positive direction of a rotation using the right-hand rule

As shown in the figure above, the thumb of the right hand points to the positive direction of the *ox* axis, the first finger points to the positive direction of the *oy* axis and the middle finger points to the positive direction of the *oz* axis. Furthermore, as shown in the figure above, in order to determine the positive direction of a rotation, the thumb of the right hand points the positive direction of the rotation axis and the direction of the bent fingers is the positive direction of rotation.

1. Coordinate System

☐ EFCF and ABCF

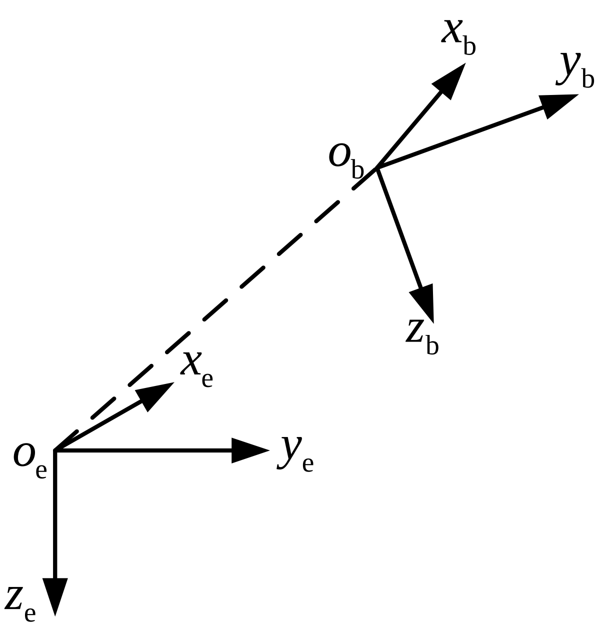


Fig 5.2 The relationship between the ABCF and the EFCF

The Earth-Fixed Coordinate Frame (EFCF) is used to study multicopter's dynamic state relative to the Earth's surface and to determine its 3D position. The Earth's curvature is ignored. The initial position of the multicopter or the center of the Earth is often set as the coordinate origin o_e, the axis $o_e x_e$ points to a certain direction in the horizontal plane and the $o_e z_e$ axis points perpendicularly to the ground. Then, the $o_e y_e$ axis is determined according to the right-hand rule.

The Aircraft-Body Coordinate Frame (ABCF) is fixed to the multicopter. The Center of Gravity (CoG) of the multicopter is chosen as the origin o_b . The $o_b x_b$ axis points to the nose direction in the symmetric plane of the multicopter. The axis $o_b z_b$ is in the symmetric plane of the multicopter, pointing downwards, perpendicular to the $o_b x_b$ axis. The $o_b y_b$ axis is determined according to the right-hand rule.

Subscript e represents the Earth, subscript b represents the Body.

1. Coordinate System

EFCF and ABCF

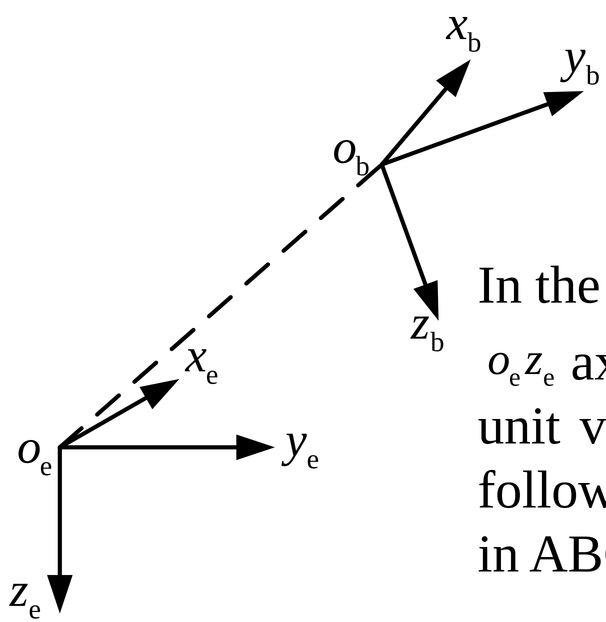


Fig 5.2 The relationship between the ABCF and the EFCF

Define the following unit vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

In the EFCF, the unit vectors along the $o_\mathrm{e}x_\mathrm{e}$ axis, $o_\mathrm{e}y_\mathrm{e}$ axis and $o_\mathrm{e}z_\mathrm{e}$ axis are expressed as $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, respectively. In the ABCF, the unit vectors along the $o_\mathrm{b}x_\mathrm{b}$ axis, $o_\mathrm{b}y_\mathrm{b}$ axis and $o_\mathrm{b}z_\mathrm{b}$ axis satisfy the following relationship (Superscript b represents the expression in ABCF of a vector)

$${}^\mathrm{b}\mathbf{b}_1 = \mathbf{e}_1, {}^\mathrm{b}\mathbf{b}_2 = \mathbf{e}_2, {}^\mathrm{b}\mathbf{b}_3 = \mathbf{e}_3$$

In the EFCF, the unit vectors along the $o_\mathrm{b}x_\mathrm{b}$ axis, $o_\mathrm{b}y_\mathrm{b}$ axis and $o_\mathrm{b}z_\mathrm{b}$ axis are expressed as ${}^\mathrm{e}\mathbf{b}_1, {}^\mathrm{e}\mathbf{b}_2, {}^\mathrm{e}\mathbf{b}_3$ respectively. (Superscript e represents the expression in EFCF of a vector).

Euler Angles

(1) Definition

The rotation from the EFCF to the ABCF is composed of three elemental rotations about $\mathbf{e}_3, \mathbf{k}_2, \mathbf{n}_1$ axes by ψ, θ, ϕ separately.

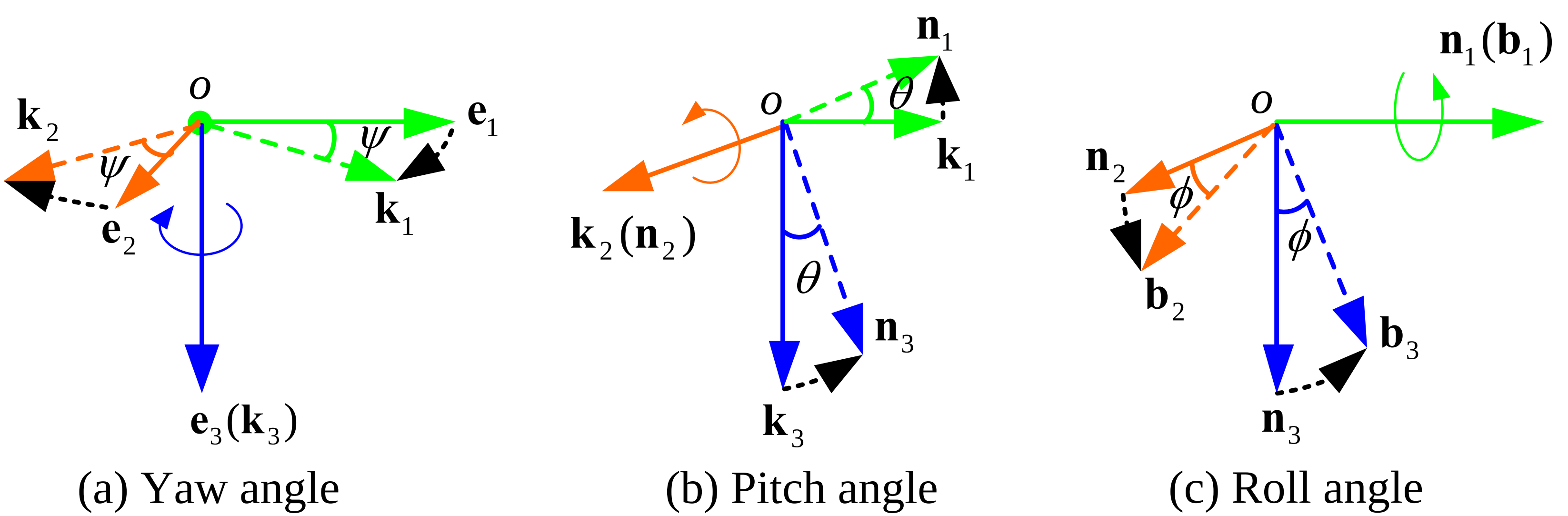


Fig 5.3 Euler angles and frame transformation

2. Attitude Representation

□ Euler Angles

(1) Definition

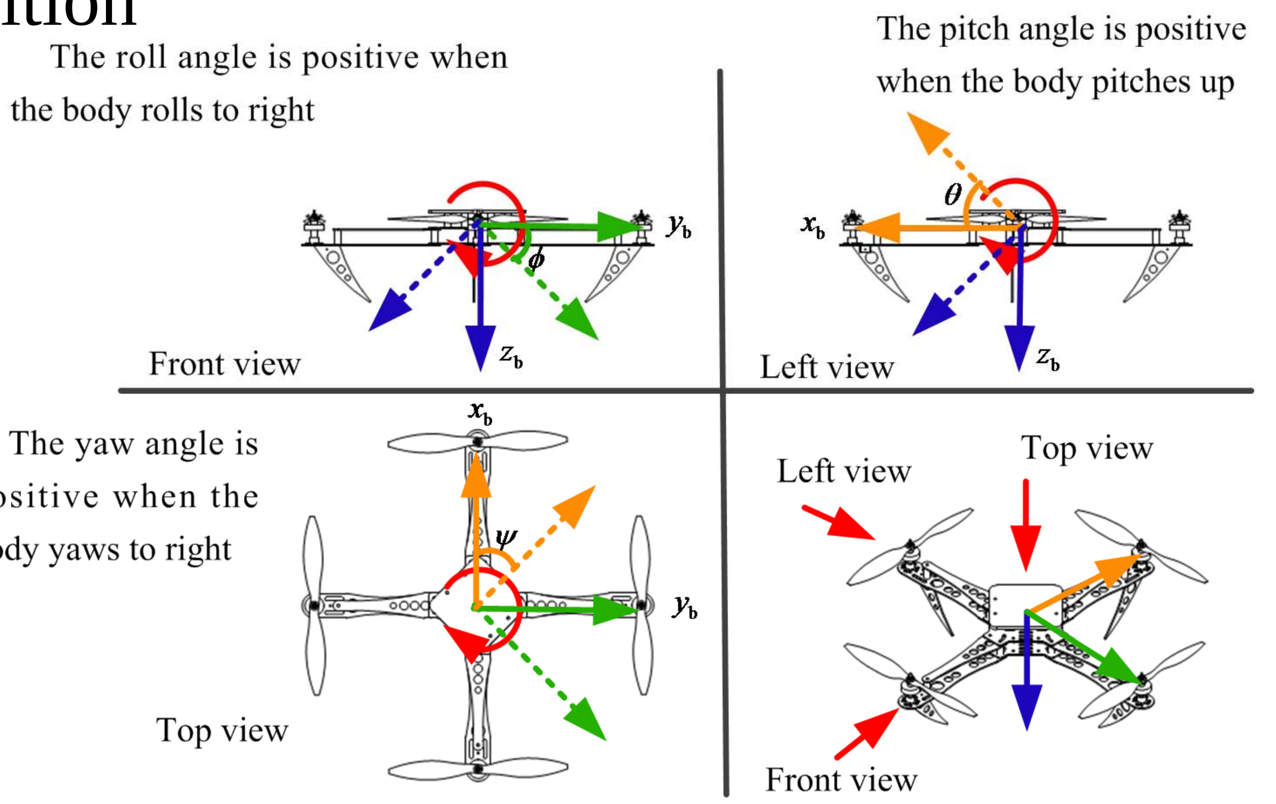


Fig 5.4 Intuitive representation of the Euler angles (*x* axis is orange , *y* axis is green, z axis is blue)

2. Attitude Representation

□ Euler Angles

(1) Definition

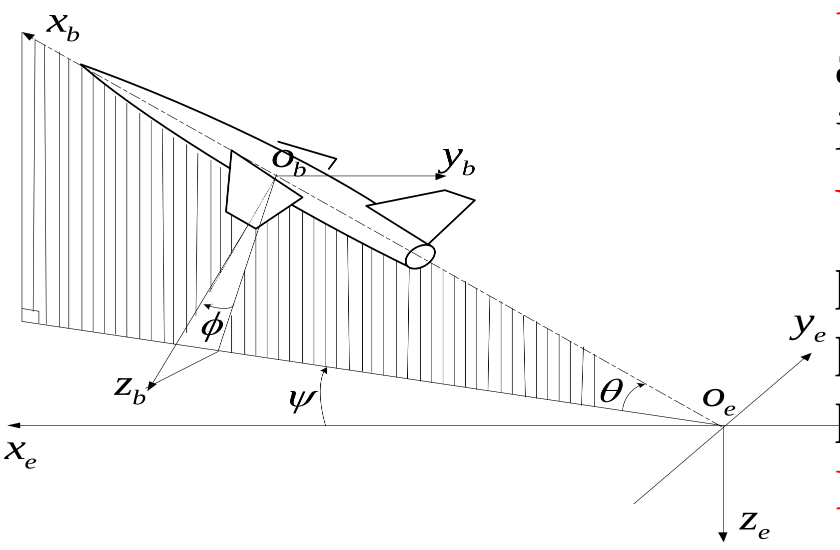


Fig 5.5 Representation of Euler angles

The angles between the EFCF and the ABCF are attitude angles, namely Euler angles.

Pitch angle θ: the angle between the body axis and the horizon plane. The pitch angle is positive when the aircraft nose pitches up.

Yaw angle ψ: the angle between the projection of the body axis in the horizon plane and the earth's axis. The yaw angle is positive when the aircraft body turns to right.

Roll angle ϕ: the rotation angle of the aircraft symmetry plane around the body axis. The roll angle is positive when the aircraft body rolls to right.

2. Attitude Representation

Euler Angles

(1) Definition

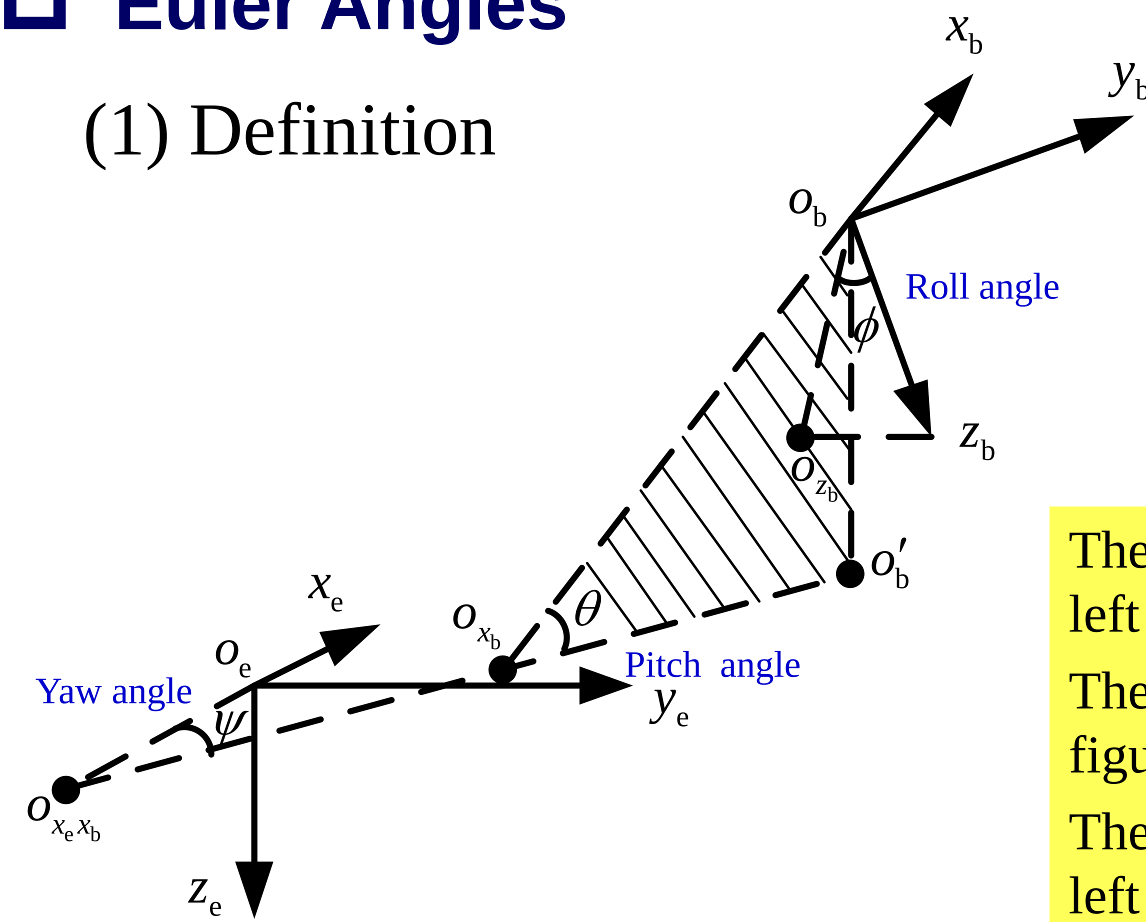


Fig 5.6 Diagram of the Euler angles

The pitch angle θ shown in the left figure is positive;

The roll angle ϕ shown in the left figure is negative;

The yaw angle ψ shown in the left figure is positive.

2. Attitude Representation

Euler Angles

(2) Relationship between the attitude rate and the aircraft body's angular velocity

The angular velocity of the aircraft body's rotation is

$$^{\mathrm{b}}\boldsymbol{\omega} = \begin{bmatrix} \omega_{x_\mathrm{b}} & \omega_{y_\mathrm{b}} & \omega_{z_\mathrm{b}} \end{bmatrix}^{\mathrm{T}}$$

Then

$$\begin{bmatrix} \omega_{x_\mathrm{b}} \\ \omega_{y_\mathrm{b}} \\ \omega_{z_\mathrm{b}} \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \; ?$$

2. Attitude Representation

Euler Angles

(2) Relationship between the attitude rate and the aircraft body's angular velocity

The angular velocity of the aircraft body's rotation is

$$^{\mathrm{b}}\boldsymbol{\omega} = \begin{bmatrix} \omega_{x_{\mathrm{b}}} & \omega_{y_{\mathrm{b}}} & \omega_{z_{\mathrm{b}}} \end{bmatrix}^{\mathrm{T}}$$

$$^{\mathrm{b}}\boldsymbol{\omega} = \dot{\psi} \cdot {}^{\mathrm{b}}\mathbf{k}_3 + \dot{\theta} \cdot {}^{\mathrm{b}}\mathbf{n}_2 + \dot{\phi} \cdot {}^{\mathrm{b}}\mathbf{b}_1$$

Superscript b represents the expression in ABCF of a vector.

Then

$$\begin{bmatrix} \omega_{x_{\mathrm{b}}} \\ \omega_{y_{\mathrm{b}}} \\ \omega_{z_{\mathrm{b}}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

2. Attitude Representation

Euler Angles

(2) Relationship between the attitude rate and the aircraft body's angular velocity

Furthermore $\dot{\mathbf{\Theta}} = \mathbf{W} \cdot {}^{\mathrm{b}}\boldsymbol{\omega}$

where $\mathbf{\Theta} = [\phi \quad \theta \quad \psi]^{\mathrm{T}}$ $\mathbf{W} = \begin{bmatrix} 1 & \tan\theta\sin\phi & \tan\theta\cos\phi \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix}$

$\theta = \pm\pi/2$

Singularity problem

When $\theta, \phi \approx 0$, one has

$$\begin{bmatrix} \omega_{x_{\mathrm{b}}} \\ \omega_{y_{\mathrm{b}}} \\ \omega_{z_{\mathrm{b}}} \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

2. Attitude Representation

Rotation Matrix

(1) Definition

Define the rotation matrix as

Rotation matrix from the ABCF to the EFCF

$$\mathbf{R}_{\mathrm{b}}^{\mathrm{e}} = \begin{bmatrix} {}^{\mathrm{e}}\mathbf{b}_1 & {}^{\mathrm{e}}\mathbf{b}_2 & {}^{\mathrm{e}}\mathbf{b}_3 \end{bmatrix}$$

$$\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\mathbf{R}_{\mathrm{b}}^{\mathrm{eT}} = \mathbf{R}_{\mathrm{b}}^{\mathrm{eT}}\mathbf{R}_{\mathrm{b}}^{\mathrm{e}} = \mathbf{I}_3$$

$$\det\left(\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\right) = 1$$

Note: det() represents the determinant

The vectors in rotation matrix satisfy

$${}^{\mathrm{e}}\mathbf{b}_1 = \mathbf{R}_{\mathrm{b}}^{\mathrm{e}} \cdot {}^{\mathrm{b}}\mathbf{b}_1 = \mathbf{R}_{\mathrm{b}}^{\mathrm{e}} \cdot \mathbf{e}_1,\ {}^{\mathrm{e}}\mathbf{b}_2 = \mathbf{R}_{\mathrm{b}}^{\mathrm{e}} \cdot {}^{\mathrm{b}}\mathbf{b}_2 = \mathbf{R}_{\mathrm{b}}^{\mathrm{e}} \cdot \mathbf{e}_2,\ {}^{\mathrm{e}}\mathbf{b}_3 = \mathbf{R}_{\mathrm{b}}^{\mathrm{e}} \cdot {}^{\mathrm{b}}\mathbf{b}_3 = \mathbf{R}_{\mathrm{b}}^{\mathrm{e}} \cdot \mathbf{e}_3$$

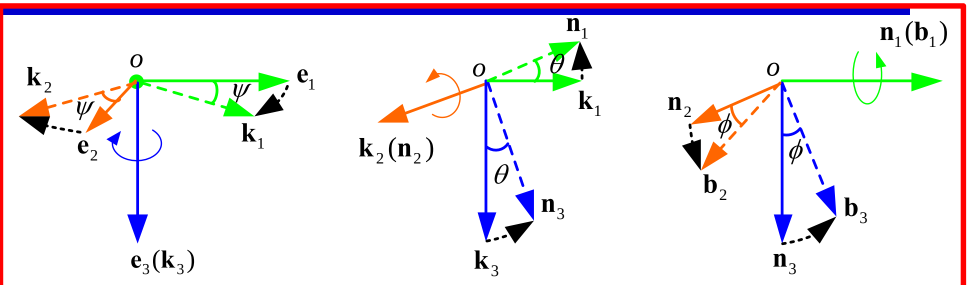
Superscript e represents the expression in EFCF of a vector.

Superscript b represents the expression in ABCF of a vector.

2. Attitude Representation

□ Rotation Matrix

(1) Definition



The rotation from the EFCF to the ABCF is composed of three elemental steps

$$\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{bmatrix} \xrightarrow{\mathbf{R}_z(\psi)} \begin{bmatrix}\mathbf{k}_1\\ \mathbf{k}_2\\ \mathbf{k}_3=\mathbf{e}_3\end{bmatrix} \xrightarrow{\mathbf{R}_y(\theta)} \begin{bmatrix}\mathbf{n}_1\\ \mathbf{n}_2=\mathbf{k}_2\\ \mathbf{n}_3\end{bmatrix} \xrightarrow{\mathbf{R}_x(\phi)} \begin{bmatrix}\mathbf{b}_1=\mathbf{n}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{bmatrix}$$

where

$$\mathbf{R}_z(\psi) \triangleq \begin{bmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}, \mathbf{R}_y(\theta) \triangleq \begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}, \mathbf{R}_x(\phi) \triangleq \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{bmatrix}.$$

2. Attitude Representation

□ Rotation Matrix

(1) Definition

$$\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{bmatrix}\xrightarrow{\mathbf{R}_z(\psi)}\begin{bmatrix}\mathbf{k}_1\\ \mathbf{k}_2\\ \mathbf{k}_3=\mathbf{e}_3\end{bmatrix}\xrightarrow{\mathbf{R}_y(\theta)}\begin{bmatrix}\mathbf{n}_1\\ \mathbf{n}_2=\mathbf{k}_2\\ \mathbf{n}_3\end{bmatrix}\xrightarrow{\mathbf{R}_x(\phi)}\begin{bmatrix}\mathbf{b}_1=\mathbf{n}_1\\ \mathbf{b}_2\\ \mathbf{b}_3\end{bmatrix}$$

$$\begin{aligned}\mathbf{R}_\mathrm{b}^\mathrm{e} &= \left(\mathbf{R}_\mathrm{e}^\mathrm{b}\right)^{-1}\\ &= \mathbf{R}_\mathrm{z}^{-1}(\psi)\mathbf{R}_y^{-1}(\theta)\mathbf{R}_x^{-1}(\phi)\\ &= \mathbf{R}_z(-\psi)\mathbf{R}_y(-\theta)\mathbf{R}_x(-\phi)\\ &= \begin{bmatrix}\cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi-\sin\psi\cos\phi & \cos\psi\sin\theta\cos\phi+\sin\psi\sin\phi\\ \cos\theta\sin\psi & \sin\psi\sin\theta\sin\phi+\cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi-\cos\psi\sin\phi\\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta\end{bmatrix}.\end{aligned}$$

2. Attitude Representation

Rotation Matrix

(1) Definition

Calculate Euler angles from the rotation matrix

$$\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}=\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\tan(\psi)=\frac{r_{21}}{r_{11}} \qquad \psi=\arctan\frac{r_{21}}{r_{11}}$$

$$\sin(\theta)=-r_{31} \qquad \theta=\arcsin(-r_{31})$$

$$\tan(\phi)=\frac{r_{32}}{r_{33}} \qquad \phi=\arctan\frac{r_{32}}{r_{33}}$$

When $\theta=\pm\frac{\pi}{2}$,

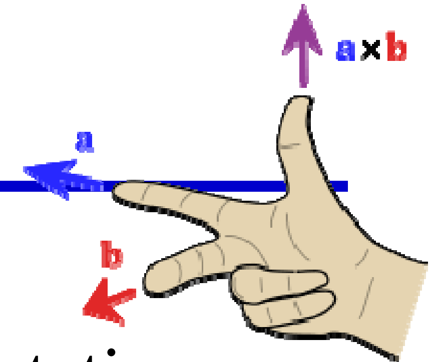
$$\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}=\begin{bmatrix} 0 & -\sin(\psi\mp\phi) & \cos(\psi\mp\phi) \\ 0 & \cos(\psi\mp\phi) & \sin(\psi\mp\phi) \\ \mp 1 & 0 & 0 \end{bmatrix}.$$

Infinite number of combinations

$\theta=\pm\pi/2$
Singularity problem

It is assumed that $\phi=0$ in this case.

2. Attitude Representation



□ Rotation Matrix

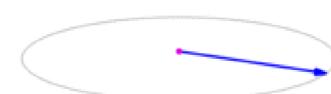
(2) Relationship between the derivative of the rotation matrix and the aircraft body's angular velocity

The cross product of two vectors $\mathbf{a} \triangleq \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{b} \triangleq \begin{bmatrix} b_x & b_y & b_z \end{bmatrix}^{\mathrm{T}}$ is defined as

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_{\times} \mathbf{b}$$

where

$$[\mathbf{a}]_{\times} \triangleq \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \qquad \mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \sin\theta \, \mathbf{n}$$



2. Attitude Representation

□ Rotation Matrix

(2) Relationship between the derivative of the rotation matrix and the aircraft body's angular velocity

If the rigid body's rotation (without translation) is only considered, then the derivative of a vector ${}^{\mathrm{e}}\mathbf{r} \in \mathbb{R}^3$ satisfies (Similar to the circular motion)

$$\frac{\mathrm{d}\,{}^{\mathrm{e}}\mathbf{r}}{\mathrm{d}t} = {}^{\mathrm{e}}\boldsymbol{\omega} \times {}^{\mathrm{e}}\mathbf{r}$$

where the symbol $\times$ represents the vector cross product. One has

$$\frac{\mathrm{d}\left[{}^{\mathrm{e}}\mathbf{b}_1 \quad {}^{\mathrm{e}}\mathbf{b}_2 \quad {}^{\mathrm{e}}\mathbf{b}_3 \right]}{\mathrm{d}t} = \left[{}^{\mathrm{e}}\boldsymbol{\omega} \times {}^{\mathrm{e}}\mathbf{b}_1 \quad {}^{\mathrm{e}}\boldsymbol{\omega} \times {}^{\mathrm{e}}\mathbf{b}_2 \quad {}^{\mathrm{e}}\boldsymbol{\omega} \times {}^{\mathrm{e}}\mathbf{b}_3 \right]$$

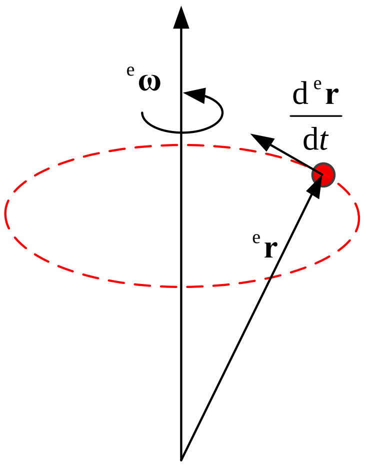


Fig 5.7 The derivative of a vector presented by the circular motion

2. Attitude Representation

☐ Rotation Matrix

(2) Relationship between the derivative of the rotation matrix and the aircraft body's angular velocity

According to ${}^{\mathrm{e}}\boldsymbol{\omega} = \mathbf{R}_{\mathrm{b}}^{\mathrm{e}} \cdot {}^{\mathrm{b}}\boldsymbol{\omega}$ and the property of the cross product

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}}{\mathrm{d}t} &= \left[\left(\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\,{}^{\mathrm{b}}\boldsymbol{\omega}\right)\times\left(\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\mathbf{e}_1\right) \quad \left(\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\,{}^{\mathrm{b}}\boldsymbol{\omega}\right)\times\left(\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\mathbf{e}_2\right) \quad \left(\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\,{}^{\mathrm{b}}\boldsymbol{\omega}\right)\times\left(\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\mathbf{e}_3\right) \right] \\
&= \left[\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\left({}^{\mathrm{b}}\boldsymbol{\omega}\times\mathbf{e}_1\right) \quad \mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\left({}^{\mathrm{b}}\boldsymbol{\omega}\times\mathbf{e}_2\right) \quad \mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\left({}^{\mathrm{b}}\boldsymbol{\omega}\times\mathbf{e}_3\right) \right] \\
&= \mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\left[{}^{\mathrm{b}}\boldsymbol{\omega}\times\mathbf{e}_1 \quad {}^{\mathrm{b}}\boldsymbol{\omega}\times\mathbf{e}_2 \quad {}^{\mathrm{b}}\boldsymbol{\omega}\times\mathbf{e}_3 \right] \\
&= \mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\left[{}^{\mathrm{b}}\boldsymbol{\omega} \right]_{\times}
\end{aligned}
$$

Following property of the cross product is used : for a rotation matrix $\mathbf{R} \in \mathbb{R}^{3\times 3}\left(\det\left(\mathbf{R}\right)=1\right)$ and any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, one has $\left(\mathbf{Ra}\right)\times\left(\mathbf{Rb}\right) = \mathbf{R}\left(\mathbf{a}\times\mathbf{b}\right)$

The use of the rotation matrix can avoid the singularity problem. However, since has nine unknown variables, the computational burden of solving equation is heavy.

2. Attitude Representation

□ Quaternion

(1) Definition

Quaternions are normally written as

$$\mathbf{q}=\begin{bmatrix} q_0 \\ \mathbf{q}_{\mathrm{v}} \end{bmatrix}$$

where $q_0 \in \mathbb{R}$ is the scalar part of $\mathbf{q} \in \mathbb{R}^4$ and $\mathbf{q}_{\mathrm{v}} = [q_1 \quad q_2 \quad q_3]^{\mathrm{T}} \in \mathbb{R}^3$ is the vector part.

For a real number $s \in \mathbb{R}$, the corresponding quaternion is defined as $\mathbf{q} = [s \quad \mathbf{0}_{1\times 3}]^{\mathrm{T}}$. For a vector $\mathbf{v} \in \mathbb{R}^3$, the corresponding quaternion is $\mathbf{q} = \begin{bmatrix} 0 & \mathbf{v}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$.

Fig 5.8 Quaternion plaque on Brougham (Broom) Bridge, Dublin, and the image is from https://en.wikipedia.org/wiki/Quaternion

It reads "Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$ & cut it on a stone of this bridge."

2. Attitude Representation

Quaternion

(2) Quaternions' basic operation rules

- Addition and subtraction

$$\mathbf{p}\pm\mathbf{q}=\begin{bmatrix}p_0\\ \mathbf{p}_{\mathrm{v}}\end{bmatrix}\pm\begin{bmatrix}q_0\\ \mathbf{q}_{\mathrm{v}}\end{bmatrix}=\begin{bmatrix}p_0\pm q_0\\ \mathbf{p}_{\mathrm{v}}\pm\mathbf{q}_{\mathrm{v}}\end{bmatrix}$$

- Multiplication

$$\mathbf{p}\otimes\mathbf{q}=\begin{bmatrix}p_0\\ \mathbf{p}_{\mathrm{v}}\end{bmatrix}\otimes\begin{bmatrix}q_0\\ \mathbf{q}_{\mathrm{v}}\end{bmatrix}=\begin{bmatrix}p_0q_0-\mathbf{q}_{\mathrm{v}}^{\mathrm{T}}\mathbf{p}_{\mathrm{v}}\\ \mathbf{p}_{\mathrm{v}}\times\mathbf{q}_{\mathrm{v}}+p_0\mathbf{q}_{\mathrm{v}}+q_0\mathbf{p}_{\mathrm{v}}\end{bmatrix}$$

Multiplication properties (Note: q, r, m are quaternions, s is a scalar, u, v are column vectors)

$$\mathbf{q}\otimes(\mathbf{r}+\mathbf{m})=\mathbf{q}\otimes\mathbf{r}+\mathbf{q}\otimes\mathbf{m}$$

$$\mathbf{q}\otimes\mathbf{r}\otimes\mathbf{m}=(\mathbf{q}\otimes\mathbf{r})\otimes\mathbf{m}=\mathbf{q}\otimes(\mathbf{r}\otimes\mathbf{m})$$

$$s\mathbf{q}=\mathbf{q}s=\begin{bmatrix}sq_0\\ s\mathbf{q}_{\mathrm{v}}\end{bmatrix}$$

$$\mathbf{q}_{\mathbf{u}}\otimes\mathbf{q}_{\mathbf{v}}=\begin{bmatrix}0\\ \mathbf{u}\end{bmatrix}\otimes\begin{bmatrix}0\\ \mathbf{v}\end{bmatrix}=\begin{bmatrix}-\mathbf{u}^{\mathrm{T}}\mathbf{v}\\ \mathbf{u}\times\mathbf{v}\end{bmatrix}$$

2. Attitude Representation

Quaternion

(2) Quaternions' basic operation rules

- Conjugate

$$\mathbf{q} = \begin{bmatrix} q_0 \\ \mathbf{q}_\mathrm{v} \end{bmatrix} \Rightarrow \mathbf{q}^* = \begin{bmatrix} q_0 \\ -\mathbf{q}_\mathrm{v} \end{bmatrix}$$

Some properties:

$$\left(\mathbf{q}^*\right)^* = \mathbf{q}$$

$$\left(\mathbf{p} \otimes \mathbf{q}\right)^* = \mathbf{q}^* \otimes \mathbf{p}^*$$

$$\left(\mathbf{p} + \mathbf{q}\right)^* = \mathbf{p}^* + \mathbf{q}^*$$

- Norm

$$\begin{aligned} \|\mathbf{q}\|^2 &= \|\mathbf{q} \otimes \mathbf{q}^*\| = \|\mathbf{q}^* \otimes \mathbf{q}\| \\ &= q_0^2 + \mathbf{q}_\mathrm{v}^\mathrm{T}\mathbf{q}_\mathrm{v} \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2 \end{aligned}$$

Some properties:

$$\|\mathbf{p} \otimes \mathbf{q}\| = \|\mathbf{p}\|\|\mathbf{q}\|$$

$$\|\mathbf{q}^*\| = \|\mathbf{q}\|$$

2. Attitude Representation

Quaternion

(2) Quaternions' basic operation rules

- Inverse

$$\mathbf{q} \otimes \mathbf{q}^{-1} = \begin{bmatrix} 1 \\ \mathbf{0}_{3\times1} \end{bmatrix}$$

According to the definition of $\mathbf{q}^*$, one has $\mathbf{q}^{-1} = \dfrac{\mathbf{q}^*}{\|\mathbf{q}\|}$

- Unit quaternion

For a unit quaternion $\mathbf{q}$, it satisfies $\|\mathbf{q}\| = 1$. Let $\|\mathbf{p}\| = \|\mathbf{q}\| = 1$. Then

$$\|\mathbf{p} \otimes \mathbf{q}\| = 1$$

$$\mathbf{q}^{-1} = \mathbf{q}^*$$

2. Attitude Representation

□ Quaternion

(3) Quaternions as rotations

Assume that $\mathbf{q}$ represents a rotation process and $\mathbf{v}_1 \in \mathbb{R}^3$ represents a vector. Then under the action of $\mathbf{q}$, the vector $\mathbf{v}_1$ is turned into $\mathbf{v}_1' \in \mathbb{R}^3$. This process is expressed as

$$\begin{bmatrix} 0 \\ \mathbf{v}_1' \end{bmatrix} = \mathbf{q} \otimes \begin{bmatrix} 0 \\ \mathbf{v}_1 \end{bmatrix} \otimes \mathbf{q}^{-1}$$

The first row always stands

A unit quaternion can always be written in the form

$$\mathbf{q} = \begin{bmatrix} \cos\dfrac{\theta}{2} \\ \mathbf{v}\sin\dfrac{\theta}{2} \end{bmatrix}$$

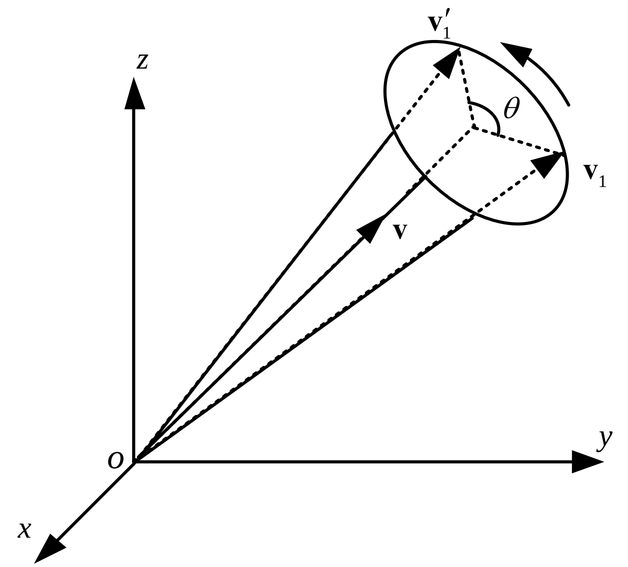


Fig 5.9 Physical meaning of unit quaternions

Readers can further refer to:

[1] Shoemake K. Quaternions. Department of Computer and Information Science, University of Pennsylvania, USA, 1994

2. Attitude Representation

□ Quaternion

(3) Quaternions as rotations

Define two unit vectors $\mathbf{v}_0, \mathbf{v}_1 \left(\mathbf{v}_1 \neq \pm\mathbf{v}_0\right)$ with $\theta/2$ being the angle between them. Therefore, one has

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{v}_1 = \cos\frac{\theta}{2}$$

$$\mathbf{v} = \frac{\mathbf{v}_0 \times \mathbf{v}_1}{\left\|\mathbf{v}_0 \times \mathbf{v}_1\right\|} = \frac{\mathbf{v}_0 \times \mathbf{v}_1}{\left\|\mathbf{v}_0\right\|\left\|\mathbf{v}_1\right\|\sin\frac{\theta}{2}} = \frac{\mathbf{v}_0 \times \mathbf{v}_1}{\sin\frac{\theta}{2}} \Rightarrow \mathbf{v}_0 \times \mathbf{v}_1 = \mathbf{v}\sin\frac{\theta}{2}$$

Define a unit quaternion, one has

$$\mathbf{q} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \mathbf{v}\sin\frac{\theta}{2} \end{bmatrix} = \begin{bmatrix} \mathbf{v}_0^{\mathrm{T}}\mathbf{v}_1 \\ \mathbf{v}_0 \times \mathbf{v}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{v}_1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \mathbf{v}_0 \end{bmatrix}^*$$

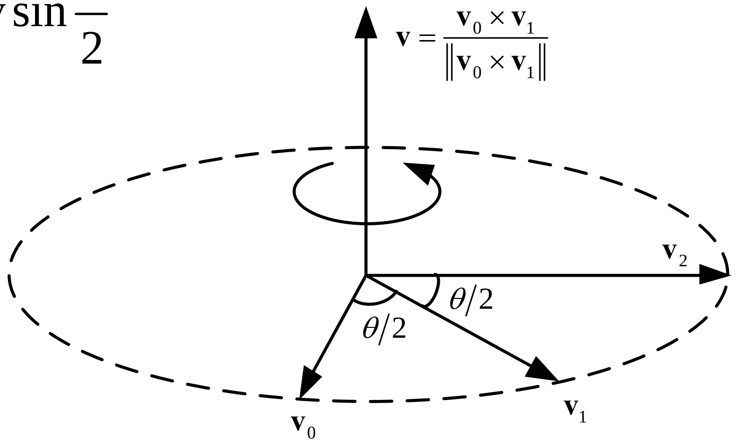


Fig 5.10 Rotation represented by quaternions

Quaternion

Why can quaternions represent the rotation?

(3) Quaternions as rotations

$$\begin{bmatrix}0\\ \mathbf{v}_2\end{bmatrix}=\mathbf{q}\otimes\begin{bmatrix}0\\ \mathbf{v}_0\end{bmatrix}\otimes\mathbf{q}^{-1}$$

$\mathbf{v}_2$ lies in the same plane as $\mathbf{v}_0$ and $\mathbf{v}_1$, as shown in the figure below, and also forms an angle $\theta/2$ with $\mathbf{v}_1$.

$$\begin{aligned}
\begin{bmatrix}0\\ \mathbf{v}_2\end{bmatrix}\otimes\begin{bmatrix}0\\ \mathbf{v}_1\end{bmatrix}^* &= \left(\mathbf{q}\otimes\begin{bmatrix}0\\ \mathbf{v}_0\end{bmatrix}\otimes\mathbf{q}^{-1}\right)\otimes\begin{bmatrix}0\\ \mathbf{v}_1\end{bmatrix}^*\\
&=\mathbf{q}\otimes\begin{bmatrix}0\\ \mathbf{v}_0\end{bmatrix}\otimes\left(\begin{bmatrix}0\\ \mathbf{v}_0\end{bmatrix}\otimes\begin{bmatrix}0\\ \mathbf{v}_1\end{bmatrix}^*\right)\otimes\begin{bmatrix}0\\ \mathbf{v}_1\end{bmatrix}^*\\
&=\mathbf{q}\otimes\left(\begin{bmatrix}0\\ \mathbf{v}_0\end{bmatrix}\otimes\begin{bmatrix}0\\ \mathbf{v}_0\end{bmatrix}\right)\otimes\left(\begin{bmatrix}0\\ \mathbf{v}_1\end{bmatrix}^*\otimes\begin{bmatrix}0\\ \mathbf{v}_1\end{bmatrix}^*\right)\\
&=\mathbf{q}\otimes\begin{bmatrix}-1\\ \mathbf{0}_{3\times1}\end{bmatrix}\otimes\begin{bmatrix}-1\\ \mathbf{0}_{3\times1}\end{bmatrix}\\
&=\mathbf{q}\\
&=\begin{bmatrix}0\\ \mathbf{v}_1\end{bmatrix}\otimes\begin{bmatrix}0\\ \mathbf{v}_0\end{bmatrix}^*
\end{aligned}$$

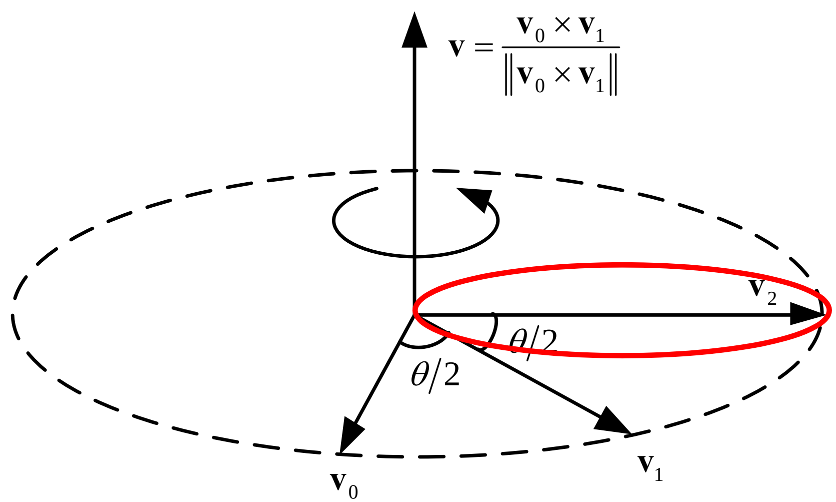


Fig 5.10 Rotation represented by quaternions

2. Attitude Representation

Quaternion

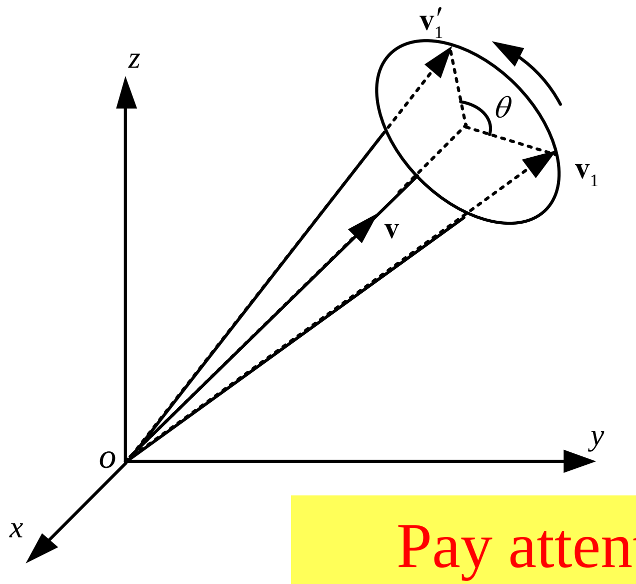
(3) Quaternions as rotations

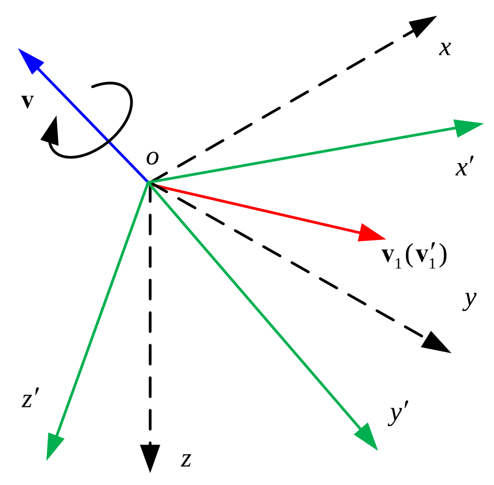
- Vector rotation

$$\begin{bmatrix} 0 \\ \mathbf{v}_1' \end{bmatrix} = \mathbf{q} \otimes \begin{bmatrix} 0 \\ \mathbf{v}_1 \end{bmatrix} \otimes \mathbf{q}^{-1}$$

- Coordinate frame rotation

$$\begin{bmatrix} 0 \\ \mathbf{v}_1' \end{bmatrix} = \mathbf{q}^{-1} \otimes \begin{bmatrix} 0 \\ \mathbf{v}_1 \end{bmatrix} \otimes \mathbf{q}$$





Pay attention to the difference !

2. Attitude Representation

□ Quaternion

(4) Quaternions and rotation matrix

It is assumed that the rotation from the EFCF to the ABCF is represented by the quaternion $\mathbf{q}_{\mathrm{e}}^{\mathrm{b}} = [q_0 \quad q_1 \quad q_2 \quad q_3]^{\mathrm{T}}$, one has (Coordinate frame rotation)

$$\begin{bmatrix} 0 \\ {}^{\mathrm{e}}\mathbf{r} \end{bmatrix} = \left(\mathbf{q}_{\mathrm{b}}^{\mathrm{e}}\right)^{-1} \otimes \begin{bmatrix} 0 \\ {}^{\mathrm{b}}\mathbf{r} \end{bmatrix} \otimes \mathbf{q}_{\mathrm{b}}^{\mathrm{e}} = \mathbf{q}_{\mathrm{e}}^{\mathrm{b}} \otimes \begin{bmatrix} 0 \\ {}^{\mathrm{b}}\mathbf{r} \end{bmatrix} \otimes \left(\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}\right)^{-1}$$

$$\Rightarrow \begin{bmatrix} 0 \\ {}^{\mathrm{e}}\mathbf{r} \end{bmatrix} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} 0 \\ {}^{\mathrm{b}}\mathbf{r} \end{bmatrix}$$

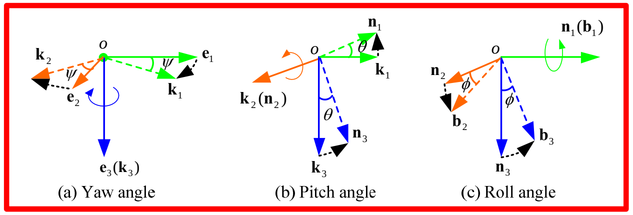
$$\Rightarrow {}^{\mathrm{e}}\mathbf{r} = \mathbf{C}(\mathbf{q}_{\mathrm{e}}^{\mathrm{b}})\,{}^{\mathrm{b}}\mathbf{r} \quad \mathbf{C}(\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}) = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

$$\Rightarrow \mathbf{R}_{\mathrm{b}}^{\mathrm{e}} = \mathbf{C}(\mathbf{q}_{\mathrm{e}}^{\mathrm{b}})$$

2. Attitude Representation

□ Quaternion

(5) Quaternions and Euler angles



According to the rotation order of Euler angles, one has

$$\begin{bmatrix} 0 \\ {}^{\mathrm{b}}\mathbf{r} \end{bmatrix} = \left(\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}\right)^{-1} \otimes \begin{bmatrix} 0 \\ {}^{\mathrm{e}}\mathbf{r} \end{bmatrix} \otimes \mathbf{q}_{\mathrm{e}}^{\mathrm{b}}$$

$$= \left(\mathbf{q}_z(\psi) \otimes \mathbf{q}_y(\theta) \otimes \mathbf{q}_x(\phi)\right)^{-1} \otimes \begin{bmatrix} 0 \\ {}^{\mathrm{e}}\mathbf{r} \end{bmatrix} \otimes \left(\mathbf{q}_z(\psi) \otimes \mathbf{q}_y(\theta) \otimes \mathbf{q}_x(\phi)\right)$$

$$= \left(\mathbf{q}_x(\phi)\right)^{-1} \otimes \left(\left(\mathbf{q}_y(\theta)\right)^{-1} \otimes \left(\left(\mathbf{q}_z(\psi)\right)^{-1} \otimes \begin{bmatrix} 0 \\ {}^{\mathrm{e}}\mathbf{r} \end{bmatrix} \otimes \mathbf{q}_z(\psi) \right) \otimes \mathbf{q}_y(\theta) \right) \otimes \mathbf{q}_x(\phi)$$

$$\mathbf{q}_{\mathrm{e}}^{\mathrm{b}} = \mathbf{q}_z(\psi) \otimes \mathbf{q}_y(\theta) \otimes \mathbf{q}_x(\phi)$$

$$\mathbf{q}_x(\phi) = \begin{bmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{q}_y(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & 0 & \sin\frac{\theta}{2} & 0 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{q}_z(\psi) = \begin{bmatrix} \cos\frac{\psi}{2} & 0 & 0 & \sin\frac{\psi}{2} \end{bmatrix}^{\mathrm{T}}$$

$$\Rightarrow$$

$$\mathbf{q}_{\mathrm{e}}^{\mathrm{b}} = \begin{bmatrix} \cos\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \cos\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} + \sin\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} - \sin\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} \end{bmatrix}$$

2. Attitude Representation

□ Quaternion

(5) Quaternions and Euler angles

$$\mathbf{q}_{\mathrm{e}}^{\mathrm{b}} = \begin{bmatrix} \cos\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}+\sin\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \sin\frac{\phi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-\cos\frac{\phi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}+\sin\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\phi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}-\sin\frac{\phi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} \end{bmatrix}$$

$$\Rightarrow \quad \tan(\phi) = \frac{2(q_0q_1+q_2q_3)}{1-2\left(q_1^2+q_2^2\right)}, \quad \sin(\theta) = 2\left(q_0q_2-q_1q_3\right), \quad \tan(\psi) = \frac{2\left(q_0q_3+q_1q_2\right)}{1-2\left(q_2^2+q_3^2\right)}$$

$$\Rightarrow \quad \phi = \arctan\frac{2(q_0q_1+q_2q_3)}{1-2\left(q_1^2+q_2^2\right)}, \quad \theta = \arcsin\left(2\left(q_0q_2-q_1q_3\right)\right), \quad \psi = \arctan\left(\frac{2\left(q_0q_3+q_1q_2\right)}{1-2\left(q_2^2+q_3^2\right)}\right)$$

When $\theta = \pm\pi/2$, the singularity problem occurs.

$$2\left(q_0q_2-q_1q_3\right)=1 \,\|\, 2\left(q_0q_2-q_1q_3\right)=-1$$

$$\mathbf{q}_{\mathrm{e}}^{\mathrm{b}} = \frac{\sqrt{2}}{2}\begin{bmatrix} \cos\left(\frac{\psi}{2}\mp\frac{\phi}{2}\right) \\ \mp\sin\left(\frac{\psi}{2}\mp\frac{\phi}{2}\right) \\ \pm\cos\left(\frac{\psi}{2}\mp\frac{\phi}{2}\right) \\ \sin\left(\frac{\psi}{2}\mp\frac{\phi}{2}\right) \end{bmatrix}$$

Infinite number of combinations

It is assumed that $\phi = 0$.

2. Attitude Representation

Quaternion

(6) Relationship between the derivative of the quaternions and the aircraft body's angular velocity

According to the composite rotation quaternion in the case of rotating coordinate frames, one has

Perturbation

$$\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t+\Delta t)=\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t)\otimes\Delta\mathbf{q}$$

where

Aircraft body's angular velocity

$$\Delta\mathbf{q}=\begin{bmatrix}1 & \frac{1}{2}{}^{\mathrm{b}}\boldsymbol{\omega}^{\mathrm{T}}\Delta t\end{bmatrix}^{\mathrm{T}}$$

The derivative of $\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t)$ is obtained as

$$\begin{aligned}
\dot{\mathbf{q}}_{\mathrm{e}}^{\mathrm{b}}(t)&=\lim_{\Delta t\to 0}\frac{\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t+\Delta t)-\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t)}{\Delta t}\\
&=\lim_{\Delta t\to 0}\frac{\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t)\otimes\Delta\mathbf{q}-\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t)}{\Delta t}\\
&=\lim_{\Delta t\to 0}\frac{\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t)\otimes\begin{bmatrix}1 & \frac{1}{2}{}^{\mathrm{b}}\boldsymbol{\omega}^{\mathrm{T}}\Delta t\end{bmatrix}^{\mathrm{T}}-\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t)}{\Delta t}\\
&=\lim_{\Delta t\to 0}\frac{\left(\mathbf{I}_3+\frac{1}{2}\begin{bmatrix}0 & -{}^{\mathrm{b}}\boldsymbol{\omega}^{\mathrm{T}}\Delta t\\ {}^{\mathrm{b}}\boldsymbol{\omega}\Delta t & -\left[{}^{\mathrm{b}}\boldsymbol{\omega}\Delta t\right]_{\times}\end{bmatrix}\right)\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t)-\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t)}{\Delta t}\\
&=\frac{1}{2}\begin{bmatrix}0 & -{}^{\mathrm{b}}\boldsymbol{\omega}^{\mathrm{T}}\\ {}^{\mathrm{b}}\boldsymbol{\omega} & -\left[{}^{\mathrm{b}}\boldsymbol{\omega}\right]_{\times}\end{bmatrix}\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t)
\end{aligned}$$

2. Attitude Representation

☐ Quaternion

(6) Relationship between the derivative of the quaternions and the aircraft body's angular velocity

$$\mathbf{q}_{\mathrm{e}}^{\mathrm{b}} = \begin{bmatrix} q_0 & \mathbf{q}_{\mathrm{v}}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

$$\dot{\mathbf{q}}_{\mathrm{e}}^{\mathrm{b}}(t) = \frac{1}{2}\begin{bmatrix} \boxed{\begin{matrix} 0 & -{}^{\mathrm{b}}\boldsymbol{\omega}^{\mathrm{T}} \\ {}^{\mathrm{b}}\boldsymbol{\omega} & -\left[{}^{\mathrm{b}}\boldsymbol{\omega}\right]_{\times} \end{matrix}} \end{bmatrix} \mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t)$$

$$\dot{q}_0 = -\frac{1}{2}\mathbf{q}_{\mathrm{v}}^{\mathrm{T}} \cdot {}^{\mathrm{b}}\boldsymbol{\omega}$$

$$\dot{\mathbf{q}}_{\mathrm{v}} = \frac{1}{2}\left(q_0 \mathbf{I}_3 + [\mathbf{q}_{\mathrm{v}}]_{\times}\right) {}^{\mathrm{b}}\boldsymbol{\omega}$$

In practice, ${}^{\mathrm{b}}\boldsymbol{\omega}$ can be measured approximately by a three-axis gyroscope, then the equation above is linear!

Transformation

$$\mathbf{R}_{\mathrm{b}}^{\mathrm{e}} = \mathbf{C}(\mathbf{q}_{\mathrm{e}}^{\mathrm{b}})$$

3. Conclusion

- Relationship between the Euler angles and the aircraft body's angular velocity

$$\dot{\boldsymbol{\Theta}} = \mathbf{W}\cdot {}^{\mathrm{b}}\boldsymbol{\omega}$$

Singularity, Nonlinear

- Relationship between the rotation matrix and the aircraft body's angular velocity

$$\frac{\mathrm{d}\mathbf{R}_{\mathrm{b}}^{\mathrm{e}}}{\mathrm{d}t} = \mathbf{R}_{\mathrm{b}}^{\mathrm{e}}\left[{}^{\mathrm{b}}\boldsymbol{\omega}\right]_{\times}$$

Non-singularity,
High dimension

- Relationship between the quaternions and the aircraft body's angular velocity

$$\dot{\mathbf{q}}_{\mathrm{e}}^{\mathrm{b}}(t) = \frac{1}{2}\begin{bmatrix} 0 & -{}^{\mathrm{b}}\boldsymbol{\omega}^{\mathrm{T}} \\ {}^{\mathrm{b}}\boldsymbol{\omega} & -\left[{}^{\mathrm{b}}\boldsymbol{\omega}\right]_{\times} \end{bmatrix}\mathbf{q}_{\mathrm{e}}^{\mathrm{b}}(t)$$

Non-singularity,
Lower dimension

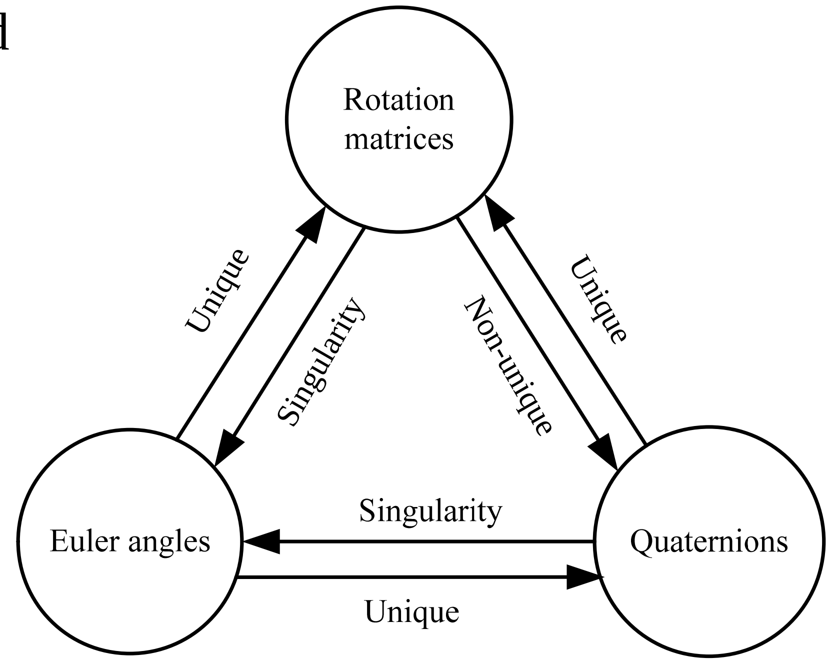


Fig 5.11 Mutual transformations among three rotation expressions

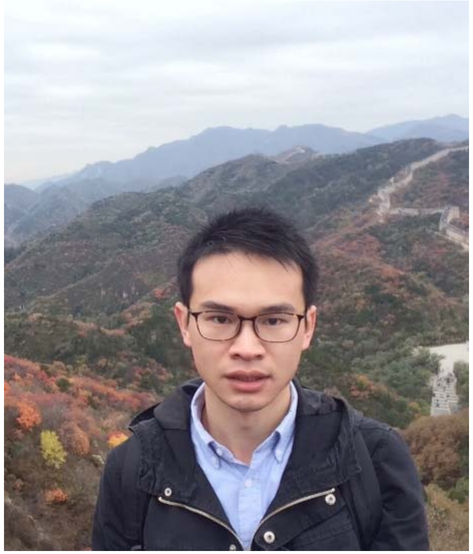
Most autopilots of multicopters use this representation.

Acknowledgement

Deep thanks go to

Yangguang Cai

Jinrui Ren



Xunhua Dai

for material preparation.

Thank you!

All course PPTs and resources can be downloaded at
http://rfly.buaa.edu.cn/course

For more detailed content, please refer to the textbook:

Quan, Quan. Introduction to Multicopter Design and Control. Springer, 2017. ISBN: 978-981-10-3382-7.

It is available now, please visit http://www.springer.com/us/book/9789811033810