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Additive Decomposition and Its Applications to Internal-Model-Based Tracking

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***Abstract*— The proposed *Additive Decomposition* is a general way to decompose an original system into two simpler systems, helping designers to analyze the original problem more explicitly. To demonstrate the effectiveness of Additive Decomposition, we apply it to the internal-model-based tracking problem. By this means, the tracking problem for the original system is decomposed into two subproblems: the tracking problem for a linear time-invariant 'primary' system and the stabilization problem for a 'secondary' system. Moreover, the former system is independent of the latter. Therefore, various special tools for analyzing linear systems can be applied to the first subproblem which is helpful to the designers. Two application examples are given to illustrate the effectiveness of the proposed Additive Decomposition.**

I. INTRODUCTION

When facing a complex problem, one often decomposes it into easier subproblems and then solves them one by one, the so called "divide and conquer" strategy. To analyze systems, the original system is usually decomposed into two or more subsystems. For example, in [1], a descriptor system is decomposed into forward and backward subsystems; the quadrotor model in [2] is divided into two subsystems: a fully-actuated subsystem and an under-actuated subsystem; in the analysis of induction machine dynamics [3], the state variables are split into two sets, one having "fast" dynamics, the other "slow" dynamics; the readers may also refer to the literature on large systems where decomposition methods are often used [4],[5].

Taking system $\dot{x}(t) = F(t,x), x \in \mathbb{R}^n$ for example, the original system $\dot{x}(t) = F(t,x)$ can be decomposed into two subsystems: $\dot{x}_1(t) = f_1(t,x_1,x_2)$ and $\dot{x}_2(t) = f_2(t,x_1,x_2)$, where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, respectively. In the literature mentioned above, the two subsystems satisfy $\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ and $x = x_1 \oplus x_2$. In this paper, we propose a new decomposition method, namely *Additive Decomposition* which satisfies $n = n_1 = n_2$, $x = x_1 + x_2$. It is proved that the combination of subsystems represents the original system under consideration. Compared with the former methods, Additive Decomposition has the following salient features.

- It is easy to follow. The proof of Additive Decomposition is basic and simple and the conclusion can be used easily. Additive Decomposition will play an important role in analyzing the tracking performance later.
- It is widely applicable. Additive Decomposition gives a general way of decomposing a general original system into two subsystems. The assumptions on Additive Decomposition are not at all stringent in practice.
- It is flexible as a design tool. Additive Decomposition is a constructive method and one of the subsystems can be selected freely by the designer.

To demonstrate the effectiveness of the proposed Additive Decomposition, we apply it to the internal-model-based tracking problem. By using Additive Decomposition, the original system is decomposed into two subsystems: a linear time-invariant 'primary' system including all external signals, leaving the derived 'secondary' system free of any external signal, such as disturbances and reference signals, where the sum of the outputs yielded by the two subsystems is equal to the tracking error of the original system and the primary system is independent of the secondary system. On this account, various special tools for linear time-invariant systems, such as Laplace transformation, transfer function, and the LMI (linear matrix inequality) approach, can be applied to the primary system. This is very helpful in the analysis of the original system. Guided by this idea, we first answer a question left open in [6], namely whether theories on modified repetitive control [7] can be applied to a class of linear systems with time-varying norm-bounded uncertainties. Secondly, we provide an alternative solution to the attitude control problem in [8, pp. 74-79]. More importantly, the proposed method can be also applied to infinite-dimensional nonlinear systems and the case where the external signals are generated by infinite-dimensional linear systems. This is problematic for methods proposed in [8].

II. ADDITIVE DECOMPOSITION

A. Additive Decomposition

Consider the following system:

$$G\left(t,\dot{X},X,d\right) = 0, X(0) = X_0 \tag{1}$$

where $X \in \mathcal{D}$ and d is the external input. For simplicity, we set the initial time $t_0 = 0$. In (1), $G\left(t,\dot{X},X,d\right) = 0$ can include, for instance, ordinary differential equations, functional differential equations, difference equations and static functions.

For the system (1), we make

Assumption 1: For a given external input d, the system (1) with initial value X_0 has a unique solution X^* on $[0,\infty)$.

Under *Assumption 1*, the following lemma on Additive Decomposition will serve as our starting point in applications. We first bring in a 'primary' system having the same

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dimension as (1), according to:

$$G_p\left(t,\dot{X}_p,X_p,d_p\right)=0, X_p(0)=X_{p,0}. \quad (2)$$

From the original system (1) and the primary system (2) we derive the following 'secondary' system:

$$G\left(t,\dot{X}_p+\dot{X}_s,X_p+X_s,d\right)-G_p\left(t,\dot{X}_p,X_p,d_p\right)=0 \quad (3)$$

with initial condition $X_s(0)=X_{s,0}$, where X_p is given by the primary system (2). Now we can state

Lemma 1 (Additive Decomposition): Under *Assumption 1,* suppose X_p^* and X_s^* are the solutions of the system (2) and (3) respectively, and the initial conditions of (1), (2) and (3) satisfy

$$X_0=X_{p,0}+X_{s,0}. \quad (4)$$

Then

$$X^*=X_p^*+X_s^*. \quad (5)$$

Proof: Since X_p^* and X_s^* are the solutions of system (2) and (3), it holds that

$$G_p\left(t,\dot{X}_p^*,X_p^*,d_p\right)=0 \quad (6)$$

$$G\left(t,\dot{X}_p^*+\dot{X}_s^*,X_p^*+X_s^*,d\right) - G_p\left(t,\dot{X}_p^*,X_p^*,d_p\right)=0. \quad (7)$$

Adding (6) to (7) yields

$$G\left(t,\dot{X}_p^*+\dot{X}_s^*,X_p^*+X_s^*,d\right)=0.$$

If the initial conditions of (1), (2) and (3) satisfy (4), then $X_p^*+X_s^*$ is also the solution of the system (1) with initial value X_0. By the uniqueness of solutions (see *Assumption 1*), the lemma follows. ■

Remark 1: In the proof above, neither system (2) nor system (3) need have a unique solution on $[0,\infty)$.

Consider the following system

$$\dot{X}(t)=F(t,X,d), X(0)=X_0. \quad (8)$$

For the system (8), we make

Assumption 2: For a given d, the system (8) with initial value X_0 has a unique solution X^* on $[0,\infty)$.

Two systems, denoted by the primary system and (derived) secondary system respectively, are defined as follows:

$$\dot{X}_p(t)=F_p(t,X_p,d_p), X_p(0)=X_{p,0} \quad (9)$$

and

$$\begin{aligned}&\dot{X}_s(t)=F(t,X_p+X_s,d)-F_p(t,X_p,d_p),\\ &X_s(0)=X_{s,0}.\end{aligned} \quad (10)$$

The secondary system (10) is determined by the original system (8) and the primary system (9).

Under *Assumption 2, Additive Decomposition Lemma* accordingly reduces to:

Corollary 1: Under *Assumption 2,* suppose X_p^* and X_s^* are the solutions of the system (9) and (10) respectively; moreover, the initial conditions of (8), (9) and (10) satisfy $X_0=X_{p,0}+X_{s,0}$. Then $X^*=X_p^*+X_s^*$.

Remark 2: By Additive Decomposition, system (1) or (8) is decomposed into two subsystems with the same dimension as the original system.

Remark 3: Neither *Assumption 1* nor *Assumption 2* are especially stringent; readers may refer to the literature on differential equations and functional differential equations for the uniqueness of solutions.

B. Examples

As seen above, Additive Decomposition is in fact a constructive method and how to choose the primary system depends on the concrete problem. In order to demonstrate Additive Decomposition explicitly, we provide the following two examples.

Example 1 (Linear Time-varying System):
Consider the linear time-varying system:

$$\left\{\begin{array}{ll}\dot{x}(t) & =[A+\Delta A(t)]x(t)\\ & \quad +A_d x(t-T)+Br(t)\\ e(t) & =-[C+\Delta C(t)]x(t)+r(t)\\ x(\theta) & =\varphi(\theta),\theta\in[-T,0]\end{array}\right. \quad (11)$$

where $e(t)$ is a tracking error, $r(t)$ is a reference signal and $\varphi(t)$ is a bounded vector valued function representing the initial condition function, $\Delta A(t)$ and $\Delta C(t)$ are time-varying norm-bounded uncertainties. The vectors and matrices in (11) are compatibly dimensioned. The system (11) satisfies *Assumptions 1-2*.

To apply Additive Decomposition to (11), choose the primary system to be a linear time-invariant system as follows:

$$\left\{\begin{array}{ll}\dot{x}_p(t) & =Ax_p(t)+A_d x_p(t-T)+Br(t)\\ e_p(t) & =-Cx_p(t)+r(t)\\ x_p(\theta) & =\varphi(\theta),\theta\in[-T,0]\end{array}\right. . \quad (12)$$

Then the secondary system is determined by the rule (3):

$$\left\{\begin{array}{ll}\dot{x}_s(t) & =[A+\Delta A(t)][x_p(t)+x_s(t)]\\ & \quad +A_d[x_p(t-T)+x_s(t-T)]+Br(t)\\ & \quad -[Ax_p(t)+A_d x_p(t-T)+Br(t)]\\ e_s(t) & =-[C+\Delta C(t)][x_p(t)+x_s(t)]+r(t)\\ & \quad -[-Cx_p(t)+r(t)]\\ x_s(\theta) & =0,\theta\in[-T,0]\end{array}\right. . \quad (13)$$

Re-arranging terms in (13), we get

$$\left\{\begin{array}{ll}\dot{x}_s(t) & =[A+\Delta A(t)]x_s(t)+A_d x_s(t-T)\\ & \quad +\Delta A(t)x_p(t)\\ e_s(t) & =-[C+\Delta C(t)]x_s(t)-\Delta C(t)x_p(t)\\ x_s(\theta) & =0,\theta\in[-T,0]\end{array}\right. . \quad (14)$$

By Additive Decomposition Lemma, $e(t)=e_p(t)+e_s(t)$. Note that (12) is a linear time-invariant system and is independent of the secondary system (14), for the analysis of which we have many tools such as the transfer function. By contrast, the transfer function tool cannot be directly applied to the original system (11) as it is time-varying.

Remark 4: In practice, neither $e_p(t)$ nor $e_s(t)$ have clear physical meanings. However, $e_p(t) + e_s(t)$ represents the tracking error. Since $\|e(t)\| \leq \|e_p(t)\| + \|e_s(t)\|$, we can analyze the tracking error $e(t)$ by analyzing $e_p(t)$ and $e_s(t)$ separately. If $e_p(t)$ and $e_s(t)$ are bounded and small, then so is $e(t)$.

Example 2 (Nonlinear System):

Consider the following nonlinear system:

$$\left\{\begin{aligned} E_1\dot{\zeta}_1(t) &= S_1(\zeta_{1,t}) + K_1(x_t) \\ E_2\dot{\zeta}_2(t) &= S_2(\zeta_{2,t}) + K_2(x_t) \\ \dot{\mu}(t) &= A_2(\mu_t) + C_2\zeta_2(t) + K_3(x_t) \\ \dot{z}(t) &= h(x_t, z_t) + C_2 w_2(t) \\ \dot{x}(t) &= f(x_t, z_t) + C_1 w_1(t) \\ &\quad - [A_1(\mu_t) + C_1\zeta_1(t)] \end{aligned}\right. \tag{15}$$

with initial conditions

$$\begin{aligned} &\zeta_i(\theta) = 0, \theta \in [-r_i, 0], i = 1, 2, \\ &\mu(\theta) = 0, \theta \in [-\max(r_1, r_2), 0], \\ &x(\theta) = \varphi_1(\theta), \theta \in [-\tau_1, 0], z(\theta) = \varphi_2(\theta), \theta \in [-\tau_2, 0] \end{aligned}$$

where $g_t \triangleq g(t+\theta)$, $\theta \in [-\tau, 0]$, and $A_1(\cdot), A_2(\cdot)$ are linear functionals. *Assumption 2* is supposed to be satisfied for (15).

The disturbances $w_1(t)$ and $w_2(t)$ affecting this system are generated by the following linear systems

$$\begin{aligned} E_i\dot{w}_i(t) &= S_1(w_{i,t}) \\ w_i(\theta) &= \phi_i(\theta), \theta \in [-r_i, 0], i = 1, 2, \end{aligned} \tag{16}$$

where $S_1(\cdot), S_2(\cdot)$ are known linear functionals, and $w_1(t), w_2(t)$ are bounded.

To apply Additive Decomposition, we choose the primary system to be a linear system as follows:

$$\left\{\begin{aligned} E_1\dot{\zeta}_{1p}(t) &= S_1(\zeta_{1p,t}) \\ E_2\dot{\zeta}_{2p}(t) &= S_2(\zeta_{2p,t}) \\ \dot{\mu}_p(t) &= A_2(\mu_{p,t}) + C_2\zeta_{2p}(t) \\ \dot{z}_p(t) &= A_2(z_{p,t}) + C_2 w_2(t) \\ \dot{x}_p(t) &= A_1(z_{p,t}) + C_1 w_1(t) \\ &\quad - [A_1(\mu_{p,t}) + C_1\zeta_{1p}(t)] \end{aligned}\right. \tag{17}$$

with initial conditions

$$\begin{aligned} &\zeta_{ip}(\theta) = \phi_i(\theta), \theta \in [-r_i, 0], i = 1, 2, \\ &\mu_p(\theta) = 0, \theta \in [-\max(r_1, r_2), 0], \\ &x_p(\theta) = 0, \theta \in [-\tau_1, 0], z_p(\theta) = 0, \theta \in [-\tau_2, 0]. \end{aligned}$$

Then the secondary system is determined by the rule (10):

$$\left\{\begin{aligned} E_1\dot{\zeta}_{1s}(t) &= S_1(\zeta_{1s,t}) + K_1(x_{p,t} + x_{s,t}) \\ E_2\dot{\zeta}_{2s}(t) &= S_2(\zeta_{2s,t}) + K_2(x_{p,t} + x_{s,t}) \\ \dot{\mu}_s(t) &= A_2(\mu_{s,t}) + C_2\zeta_{2s} + K_3(x_{p,t} + x_{s,t}) \\ \dot{z}_s(t) &= h(x_{p,t} + x_{s,t}, z_{p,t} + z_{s,t}) - A_2(z_{p,t}) \\ \dot{x}_s(t) &= f(x_{p,t} + x_{s,t}, z_{p,t} + z_{s,t}) \\ &\quad - A_1(z_{p,t}) - [A_1(\mu_{s,t}) + C_1\zeta_{1s}(t)] \end{aligned}\right. \tag{18}$$

with initial conditions

$$\begin{aligned} &\zeta_{is}(\theta) = 0, \theta \in [-r_i, 0], i = 1, 2, \\ &\mu_s(\theta) = 0, \theta \in [-\max(r_1, r_2), 0], \\ &x_s(\theta) = \varphi_1(\theta), \theta \in [-\tau_1, 0], z_s(\theta) = \varphi_2(\theta), \theta \in [-\tau_2, 0]. \end{aligned}$$

Note that the initial conditions on $\zeta_{1p}(t)$ and $\zeta_{2p}(t)$ are the same as those on $w_1(t)$ and $w_2(t)$; then $\zeta_{1p}(t) \equiv w_1(t)$ and $\zeta_{2p}(t) \equiv w_2(t)$. Similarly, we can obtain $z_p(t) \equiv \mu_p(t)$. Consequently, $x_p(0) = 0$ implies $x_p(t) \equiv 0$. Then the primary system (17) reduces to

$$\left\{\begin{aligned} \dot{z}_p(t) &= A_2(z_{p,t}) + C_2 w_2(t) \\ x_p(t) &\equiv 0, \zeta_{1p}(t) \equiv w_1(t) \\ \zeta_{2p}(t) &\equiv w_2(t), \mu_p(t) \equiv z_p(t) \end{aligned}\right. \tag{19}$$

with initial condition $z_p(\theta) = 0, \theta \in [-\tau_2, 0]$. On the other hand, substituting $x_p(t) \equiv 0$ into (18) results in

$$\left\{\begin{aligned} E_1\dot{\zeta}_{1s}(t) &= S_1(\zeta_{1s,t}) + K_1(x_{st}) \\ E_2\dot{\zeta}_{2s}(t) &= S_2(\zeta_{2s,t}) + K_2(x_{st}) \\ \dot{\mu}_s(t) &= A_2(\mu_{s,t}) + C_2\zeta_{2s}(t) + K_3(x_{st}) \\ \dot{z}_s(t) &= h(x_{s,t}, z_{p,t} + z_{s,t}) - A_2(z_{p,t}) \\ \dot{x}_s(t) &= f(x_{s,t}, z_{p,t} + z_{s,t}) \\ &\quad - A_1(z_{p,t}) - [A_1(\mu_{s,t}) + C_1\zeta_{1s}(t)] \end{aligned}\right. . \tag{20}$$

By Additive Decomposition Lemma, we have $x(t) = x_s(t)$ and $z(t) = z_p(t) + z_s(t)$.

III. Additive Decomposition in the Internal-Model-Based Tracking problem

There are essentially three different approaches to the asymptotic tracking of prescribed trajectories and/or rejection of disturbances [8, pp. 1-2]: tracking by dynamic inversion, adaptive tracking, and tracking via internal models. In this paper, we show how Additive Decomposition is used in the internal-model-based tracking problem [8],[9],[10].

A. Decomposition Principle

Linear time-invariant systems are very familiar. In addition, there exist many tools to analyze them, such as Laplace transformation and transfer function, the LMI approach. Based on the above consideration, the original system is usually decomposed into two subsystems by Additive Decomposition: a linear time-invariant system including all external signals as the primary system, leaving the secondary system free of any external signal, such as disturbances and reference signals. Take (11) in *Example 1* for example. The primary system (12) is chosen to be a linear time-invariant system including all external signals, while the secondary system (14) does not include any external signal. Since all external signals are introduced into the linear time-invariant system, we have several methods to deal with this problem, i.e., the tracking problem for linear time-invariant systems. Since $e(t) = e_p(t) + e_s(t)$ by Additive Decomposition Lemma, the remaining problem is to arrange $e_s(t)$. Since the secondary system (14) does not include any external signal, this is in fact a stabilization problem. Therefore, the tracking problem of the original system can be decomposed into two subproblems by Additive Decomposition as shown in Fig.1: a tracking problem for a linear time-invariant ‘primary’ system and a stabilization problem for a ‘secondary’ system. This will be further confirmed in the following section.

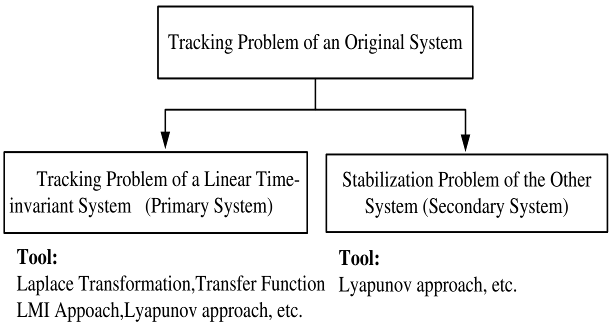

Fig. 1. Decomposition Principle

B. Application I: Modified Repetitive Controller Used in a Linear Time-varying System

Any periodic signal $r(t) \in \mathbb{R}^m$ with a period T can be generated by the free time-delay system $\frac{1}{1-e^{-sT}}I_m$ with an appropriate initial function. It is therefore expected from the internal model principle [9],[10] that the asymptotic tracking property for exogenous periodic signals may be achieved by incorporating the model $\frac{1}{1-e^{-sT}}I_m$ into the closed-loop system. Since low frequency band is dominant in any reference signal, this will virtually satisfy any practical demands. Thus, the modified repetitive controller $\frac{1}{1-q(s)e^{-sT}}I_m$ is incorporated into the closed-loop system in which the low-pass filter $q(s)$ is needed to ensure system stability. Readers may refer to [7] for information on modified repetitive control.

In [6], a modified repetitive controller is designed through an optimization problem with an LMI constraint of the free parameter. It is verified from a simulation that the designed controller improves tracking accuracy in spite of time-varying uncertainties. However, theories on modified repetitive control cannot be applied to linear time-varying systems directly, for Laplace transformation and the transfer function play an essential role in these theories. Therefore there exists a gap between linear time-invariant systems and linear time-varying systems when using the theories on modified repetitive control. In this section, we will fill this gap with the help of Additive Decomposition.

The closed-loop system considered in [6] can be represented by a state differential equation as (11) in *Example 1*. Readers may refer to [6] for the details. The reference signal $r(t)$ is a periodic signal with a period T. By Additive Decomposition, the original closed-loop system is decomposed into the primary system (12) and the secondary system (14). Because the primary system (12) is the original closed-loop system without time-varying norm-bounded uncertainties, the theories on modified repetitive control can be applied to it. Assume $\sup\limits_{t\in[0,\infty)} \|e_p(t)\| \leq \varepsilon_{e_p}$ and $\sup\limits_{t\in[0,\infty)} \|x_p(t)\| \leq \varepsilon_{x_p}$. On the other hand, it has been proven that the zero solution of the following system

$$\dot{x}(t) = [A + \Delta A(t)]\, x(t) + A_d x(t-T) \tag{21}$$

is asymptotically stable. Since $\Delta A(t)$ is bounded, it follows that the system above is globally exponentially stable. The fundamental solution of (21) satisfies $\|U(t,\xi)\| \leq Ke^{-\alpha(t-\xi)}, \alpha > 0, K > 0$, then $e_s(t)$ in (14) can be written as [11, pp. 21,145,147]:

$$\begin{aligned} e_s(t) = &-[C + \Delta C(t)] \int_0^t U(t,\xi)\, \Delta A(\xi) x_p(\xi)\, d\xi \\ &- \Delta C(t) x_p(t). \end{aligned}$$

Taking the norm $\|\cdot\|$ on both sides of the above equation yields

$$\|e_s(t)\| \leq \frac{K}{\alpha}(\|C\| + b_{\Delta C})\, \varepsilon_{x_p} b_{\Delta A} + \varepsilon_{x_p} b_{\Delta C}$$

where $b_{\Delta C} = \sup\limits_{t\in[0,\infty)} \|\Delta C(t)\|, b_{\Delta A} = \sup\limits_{t\in[0,\infty)} \|\Delta A(t)\|$. Therefore

$$\begin{aligned} \|e(t)\| &\leq \|e_p(t)\| + \|e_s(t)\| \\ &\leq \varepsilon_{e_p} + \frac{K}{\alpha}(\|C\| + b_{\Delta C})\, \varepsilon_{x_p} b_{\Delta A} + \varepsilon_{x_p} b_{\Delta C}. \end{aligned}$$

From the derivation above, the low-pass filter in the internal model still plays the role of balancing tracking performance with stability. Therefore, the modified repetitive controller can be also applied to linear time-invariant systems subject to time-varying norm-bounded uncertainties and achieves a tradeoff. The tracking error approaches $e_p(t)$, if the bound on the uncertainties is small enough, i.e., the linear time-varying system approaches a linear time-invariant system.

C. Application II: Dynamic Feedback Controller Used in a Nonlinear System

Consider the following nonlinear system

$$\begin{aligned} \dot{z}(t) &= h(x_t, z_t) + C_2 w_2(t) \\ \dot{x}(t) &= f(x_t, z_t) + u_{im}(t) + C_1 w_1(t) \end{aligned} \tag{22}$$

with initial condition

$$x(\theta) = \varphi_1(\theta), \theta \in [-\tau_1, 0], z(\theta) = \varphi_2(\theta), \theta \in [-\tau_2, 0].$$

Here $x(t), z(t)$ are the state vectors, $x(t)$ is also the regulated output, $w_1(t), w_2(t)$ are the disturbances defined in (16), $u_{im}(t)$ is the controller input used to compensate for $w_1(t), w_2(t)$; $f(\cdot)$ and $h(\cdot)$ are nonlinear functionals defined in (15). Design the controller $u_{im}(t)$ as

$$\begin{aligned} E_1\dot{\zeta}_1(t) &= S_1(\zeta_{1,t}) + K_1(x_t) \\ E_2\dot{\zeta}_2(t) &= S_2(\zeta_{2,t}) + K_2(x_t) \\ \dot{\mu}(t) &= A_2(\mu_t) + C_2\zeta_2(t) + K_3(x_t) \\ u_{im}(t) &= -[A_1(\mu_t) + C_1\zeta_1(t)] \end{aligned} \tag{23}$$

with initial condition

$$\begin{aligned} &\zeta_i(\theta) = 0, \theta \in [-r_i, 0], i = 1, 2, \\ &\mu(\theta) = 0, \theta \in [-\max(r_1, r_2), 0]. \end{aligned}$$

The closed-loop system forming by (22) and (23) is shown in (15). Based on (15), we have

Theorem 1: Suppose (i) $f(x_t, z_t)$ and $h(x_t, z_t)$ have the following forms

$$f(x_t, z_t) = f_0(x_t) + L_1(z_t)$$
$$h(x_t, z_t) = h_0(x_t) + L_2(z_t)$$

where $f_0(x_t), h_0(x_t)$ are nonlinear functionals, and $L_1(z_t), L_2(z_t)$ are linear functionals, (ii) $\dot{z}(t) = L_2(z_t)$ is globally exponentially stable, (iii) let $A_1(\cdot) = L_1(\cdot)$ and $A_2(\cdot) = L_2(\cdot)$, (iv) the solution $x(t) = 0$ of the system (15) with $w_1(t) \equiv 0, w_2(t) \equiv 0$ is globally asymptotically stable and the other variables are bounded. Then $\lim_{t\to\infty} x(t) = 0$ and $z(t)$ is bounded in the system (15).

Proof: The closed-loop system (15) can be decomposed into the primary system (19) and the secondary system (20). Since conditions (ii)-(iii) hold, $z_p(t)$ is bounded. We now consider the secondary system (20). Applying condition (i) and (iii) to (20) results in

$$\left\{\begin{array}{rl} E_1\dot{\zeta}_{1s}(t) & = S_1(\zeta_{1s,t}) + K_1(x_{s,t}) \\ E_2\dot{\zeta}_{2s}(t) & = S_2(\zeta_{2s,t}) + K_2(x_{s,t}) \\ \dot{\mu}_s(t) & = A_2(\mu_{s,t}) + C_2\zeta_{2s}(t) + K_3(x_{s,t}) \\ \dot{z}_s(t) & = h_0(x_{s,t}) + L_2(z_{s,t}) \\ \dot{x}_s(t) & = f_0(x_{s,t}) + L_1(z_{s,t}) \\ & \quad - [L_1(\mu_{s,t}) + C_1\zeta_{1s}(t)] \end{array}\right. .$$

The system above is in fact the closed-loop system (15) with $w_1(t) \equiv 0, w_2(t) \equiv 0$. Using the condition (iv), we obtain that $\lim_{t\to\infty} x_s(t) = 0$ and $z_s(t)$ is bounded. Since $x(t) = x_s(t)$ and $z(t) = z_p(t) + z_s(t)$ by Additive Decomposition Lemma (see *Example 2*), we can conclude this proof. ■

Theorem 2: Suppose (i) $w_2(t) \equiv 0$ (ii) the solution $x(t) = 0$ of the system (15) with $w_1(t) \equiv 0$ is globally asymptotically stable and the other variables are bounded. Then $\lim_{t\to\infty} x(t) = 0$ and $z(t)$ is bounded in the system (15).

Proof: The closed-loop system (15) can be decomposed into the primary system (19) and the secondary system (20). Since $w_2(t) \equiv 0$, we obtain $z_p(t) \equiv 0$ in the primary system (19). Thus, the secondary system (20) reduces to

$$\left\{\begin{array}{rl} E_1\dot{\zeta}_{1s}(t) & = S_1(\zeta_{1s,t}) + K_1(x_{s,t}) \\ E_2\dot{\zeta}_{2s}(t) & = S_2(\zeta_{2s,t}) + K_2(x_{s,t}) \\ \dot{\mu}_s(t) & = A_2(\mu_{s,t}) + C_2\zeta_{2s}(t) + K_3(x_{s,t}) \\ \dot{z}_s(t) & = h(x_{s,t}, z_{s,t}) \\ \dot{x}_s(t) & = f(x_{s,t}, z_{s,t}) \\ & \quad - [A_1(\mu_{s,t}) + C_1\zeta_{1s}(t)] \end{array}\right. . \tag{24}$$

Using the condition (ii), we obtain that $\lim_{t\to\infty} x_s(t) = 0$ and $z_s(t)$ is bounded. Since $x(t) = x_s(t)$ and $z(t) = z_p(t) + z_s(t)$ by Additive Decomposition Lemma (see *Example 2*), we can conclude this proof. ■

With *Theorem 2* in hand, we have

Corollary 2: Suppose (i) $w_2(t) \equiv 0$, (ii) the solution $x(t) = 0$ in the following system

$$\left\{\begin{array}{rl} E_1\dot{\zeta}_1(t) & = S_1(\zeta_{1,t}) + K_1(x_t) \\ \dot{z}(t) & = h(x_t, z_t) \\ \dot{x}(t) & = f(x_t, z_t) - C_1\zeta_1(t) \end{array}\right. \tag{25}$$

is globally asymptotically stable and the other variables are bounded. Then $\lim_{t\to\infty} x(t) = 0$ and $z(t)$ is bounded in the system (15).

Proof: Let $K_2(\cdot) = K_3(\cdot) = 0$, then $\zeta_2(t) \equiv 0$ and $\mu(t) \equiv 0$ in the controller (23). Consequently, the controller reduces to

$$E_1\dot{\zeta}_1(t) = S_1(\zeta_{1,t}) + K_1(x_t), u_{im}(t) = -C_1\zeta_1(t) \tag{26}$$

and the resulting closed-loop system (15) reduces to

$$\left\{\begin{array}{rl} E_1\dot{\zeta}_1(t) & = S_1(\zeta_{1,t}) + K_1(x_t) \\ \dot{z}(t) & = h(x_t, z_t) \\ \dot{x}(t) & = f(x_t, z_t) + C_1w_1(t) - C_1\zeta_1(t) \end{array}\right. .$$

The following proof is similar to that of *Theorem 2.* ■

Next, we apply the obtained results to the attitude control problem for a spacecraft operating in a low-Earth orbit.

Example 3 (Attitude Control Problem):

The attitude control problem is simplified as follows [8, pp. 74-75]:

$$\begin{array}{rl} \dot{\tilde{\mathbf{q}}}(t) & = -\frac{k_1}{2}E(\tilde{\mathbf{q}}(t))\tilde{q}(t) + \frac{1}{2}E(\tilde{\mathbf{q}}(t))x(t) \\ \dot{x}(t) & = \chi(\tilde{\mathbf{q}}(t), x(t)) + u(t) + \Gamma d(t) \end{array} . \tag{27}$$

Here $x(t) \in \mathbb{R}^3, k_1 \in \mathbb{R}^+, \tilde{\mathbf{q}} = \left[\begin{array}{cc} \tilde{q}_0 & \tilde{q}^T \end{array}\right]^T \in \mathbb{R}^4$ in which $\tilde{q}_0(t) \in \mathbb{R}$ and $\tilde{q}(t) \in \mathbb{R}^3$ denote the scalar part and vector part respectively, $E(\tilde{\mathbf{q}}(t)) \in \mathbb{R}^{4\times 3}$ is defined in [8, p. 201], $\chi(\tilde{\mathbf{q}}(t), x(t))$ denotes the nonlinear uncertainty. The control objective is to design $u(t)$ to make that $\lim_{t\to\infty} x(t) = 0$ and $\tilde{\mathbf{q}}(t)$ is bounded.

Design $u(t)$ to be $u(t) = u_{im}(t) + u_{st}(t)$, where $u_{im}(t)$ is an "internal model" controller which is used to compensate for the periodic disturbance $d(t)$, and $u_{st}(t)$ is a "stabilizing" controller which deals with the nonlinear uncertainty $\chi(\tilde{\mathbf{q}}(t), x(t))$. Then (27) can be written in the form of (22) with

$$f(x_t, z_t) = \chi(\tilde{\mathbf{q}}(t), x(t)) + u_{st}(t), z = \tilde{\mathbf{q}}$$
$$h(x_t, z_t) = -\frac{k_1}{2}E(\tilde{\mathbf{q}}(t))\tilde{q}(t) + \frac{1}{2}E(\tilde{\mathbf{q}}(t))x(t)$$
$$C_1 = \Gamma, w_1(t) = d(t), C_2 = 0, w_2(t) \equiv 0.$$

Case 1: The external torque $d(t)$ is periodic with a period T and generated by

$$\dot{d}(t) = \Phi d(t), d(0) = d_0$$

where the matrix Φ has all simple eigenvalues on the imaginary axis. In this case, according to (26), $u_{im}(t)$ is designed as

$$\dot{\zeta}_1(t) = \Phi\zeta_1(t) + K_1(x_t), u_{im}(t) = -\Gamma\zeta_1(t).$$

Through the Lyapunov approach as in [8, p. 201], if $K_1(x_t)$ and $u_{st}(t)$ are designed as

$$K_1(x_t) = \frac{1}{\gamma}P^{-1}\Gamma^T x(t), u_{st}(t) = -k_2(1 + \|x(t)\|)x(t)$$

where $\gamma, k_2 \in \mathbb{R}^+$ are chosen appropriately, and P is a positive definite solution of the Lyapunov matrix inequality

$P\Phi+\Phi^T P \leq 0$, then the solution $x(t)=0$ of the following system

$$\begin{cases} \dot{\zeta}_1(t) &= \Phi\zeta_1(t)+K_1(x_t) \\ \dot{\tilde{\mathbf{q}}}(t) &= -\frac{k_1}{2}E(\tilde{\mathbf{q}}(t))\tilde{q}(t)+\frac{1}{2}E(\tilde{\mathbf{q}}(t))x(t) \\ \dot{x}(t) &= \chi(\tilde{\mathbf{q}}(t),x(t))+u_{st}(t)-\Gamma\zeta_1(t) \end{cases}$$

is globally asymptotically stable, and $\tilde{\mathbf{q}}(t),\zeta_1(t)$ are bounded. According to *Corollary 1*, we obtain that $\lim_{t\to\infty} x(t)=0$ and $\tilde{\mathbf{q}}(t)$ is bounded when the system (27) is driven by the controller designed above.

Case 2: The external torque $d(t)$ is periodic with a period T and generated by

$$d(t)=d(t-T),d(\theta)=\phi(\theta),\theta\in[-r_1,0]. \tag{28}$$

In this case, according to (26), $u_{im}(t)$ is designed as

$$\zeta_1(t)=\zeta_1(t-T)+K_1(x_t),u_{im}(t)=-\Gamma\zeta_1(t).$$

According to *Corollary 1*, if the solution $x(t)=0$ of the following system

$$\begin{cases} \zeta_1(t) &= \zeta_1(t-T)+K_1(x_t) \\ \dot{\tilde{\mathbf{q}}}(t) &= -\frac{k_1}{2}E(\tilde{\mathbf{q}}(t))\tilde{q}(t)+\frac{1}{2}E(\tilde{\mathbf{q}}(t))x(t) \\ \dot{x}(t) &= \chi(\tilde{\mathbf{q}}(t),x(t))+u_{st}(t)-C_1\zeta_1(t) \end{cases} \tag{29}$$

is globally asymptotically stable, and $\tilde{\mathbf{q}}(t),\zeta_1(t)$ are bounded, then $\lim_{t\to\infty} x(t)=0$ and $\tilde{\mathbf{q}}(t)$ is bounded when the system (27) is driven by the controller designed above. For (29), design a Lyapunov functional

$$V(\zeta_1,\tilde{\mathbf{q}},x,t)=\frac{\gamma}{2}\int_{t-T}^{t}\zeta_1^T(\xi)\zeta_1(\xi)d\xi+(1-\tilde{q}_0)^2 + \tilde{q}^T(t)\tilde{q}(t)+\frac{1}{2}x^T(t)x(t).$$

Through the Lyapunov approach as in [8, p. 201], if $K_1(x_t)$ and $u_{st}(t)$ are designed as

$$K_1(x_t)=\frac{1}{\gamma}\Gamma^T x(t),u_{st}(t)=-k_2(1+\|z(t)\|)z(t)$$

with appropriate $\gamma,k_2\in\mathbb{R}^+$, then the solution $x(t)=0$ of (29) is globally asymptotically stable and $\tilde{\mathbf{q}}(t)$ are bounded.

Remark 5: Guided by the geometric approach, Isidori et al. in [8] have proposed internal-model-based tracking methods for both linear systems and nonlinear systems. The attitude control problem in *Case 1* is solved as an application. However, the geometric approach is only applicable to the case where the closed-loop system is finite-dimensional. When the external signals are generated by (28), the closed-loop system (29) is infinite-dimensional. This is a difficulty for the application of methods proposed in [8, pp. 74-79]. In this paper, we give an alternative solution of the attitude control problem as in [8, pp. 74-79]. More importantly, the proposed method can be also applied to infinite-dimensional nonlinear systems and the case where the external signals are generated by infinite-dimensional systems (See *Case 2*).

IV. Conclusions

In general, tracking problems are more difficult than stabilization problems, especially for nonlinear systems. By using Additive Decomposition, the internal-model-based tracking problem of the original system is decomposed into two subproblems: the tracking problem for a linear time-invariant primary system and the stabilization problem for the secondary system. On this account, frequency-domain methods and time-domain methods can be both applied no matter whether the original system is time-varying or nonlinear. This helps to make the analysis of tracking problems easier. Guided by this idea, we first obtain a conclusion that theories on modified repetitive control can be applied to a class of linear systems with time-varying norm-bounded uncertainties. Then, we propose methods of internal-model-based tracking that can be applied to infinite-dimensional nonlinear systems and the case where the external signals are generated by infinite-dimensional systems.

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