Repetitive control by output error for a class of uncertain time-delay systems

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Abstract: The problem of designing output error controllers to track periodic reference signals for a class of uncertain linear time-delay systems subject to periodic disturbances is addressed. A repetitive controller, which is a type of output error controller, is developed for these systems and a sufficient condition for stability of the closed-loop system is derived by applying the Lyapunov−Krasovskii functional and linear matrix inequalities (LMIs). In order to relax the stability condition, the theory of the modified repetitive control (RC) system is extended to the systems represented by irrational transfer functions. Based on the extended theory of the modified RC system, a modified repetitive controller, another type of output error controller, is developed for these systems and sufficient conditions for stability of the closed-loop system are derived in terms of a frequency-domain criterion and LMI. Numerical simulations are presented to demonstrate the effectiveness of the proposed controllers.

1   Introduction

It is widely known that time-delay systems exist pervasively in engineering. The evolution of these systems with time depends not only on the current state but also on past states. Time delay usually degrades system performance and more importantly, it may destroy the stability of a control system. Therefore many approaches that ensure the stabilisation of time-delay systems have been presented [1].

Recently, various methods for tracking dynamic signals of linear time-delay systems subject to unknown disturbances have been proposed. In [2], the proposed controller guaranteed uniform ultimate boundedness of the tracking error. Furthermore, the error bound could be made arbitrarily small by increasing the controller gain. In [3, 4], non-linear controllers were developed to guarantee that the outputs of the controlled uncertain time-delay system tracked the outputs of the non-delay reference model. The tracking error is made to converge to zero by increasing the maximum value of control effort. In order to achieve the

same tracking performance, it is required that the controller gain or the control effort of the non-linear controller need to be increased as the disturbance amplitude increases.

The drawbacks of the high-gain feedback solutions are related to the fact that these may saturate the joint actuators or excite high-frequency modes. Moreover, the controllers mentioned above are model-following control schemes. This implies that the reference output is from a known reference model. In practice, such reference models are often difficult to obtain. However, for a class of uncertain time-delay systems subject to periodic disturbances, we can overcome these drawbacks by developing repetitive controllers, in which a learning-based feedforward term plays a similar role as integrators play in PID controllers.

In this paper, a repetitive controller and a modified repetitive controller are proposed to track periodic reference signals for a class of uncertain linear state-delayed systems subject to periodic disturbances. To start with, an assumption that the desired trajectory is given by a periodic and bounded input is introduced. Based on this

assumption, we convert the tracking problem to a stabilisation problem for an error-driven dynamic system. Specifically, the problem is the stability of a critical case of a neutral delay equation which is difficult to solve. Under certain conditions, a sufficient condition for the stability of the repetitive control (RC) system is derived by applying the Lyapunov–Krasovskii functional and linear matrix inequality (LMI). In order to relax the condition, a modified repetitive controller is proposed to take the place of the repetitive controller. Since the transfer function of the time-delay system is not a rational transfer function, the theory of the modified RC system proposed by Hara *et al.* [5] cannot be applied directly. Therefore, we extend the theory in order to make it applicable to irrational transfer functions. Based on the extended theory, a filter is introduced into the closed-loop system to achieve a trade-off between good tracking performance and stability margin. Sufficient criteria for the stability of modified RC systems are derived in terms of a frequency-domain criterion and an LMI.

The notation used in this paper is as follows. The period T is known a priori. $\mathbb{R}^n$ is the Euclidean space of dimension n. $\boldsymbol{I}_n$ is the identity matrix with dimension n. $\boldsymbol{X} > 0\,(\boldsymbol{X} < 0)$ denotes matrix $\boldsymbol{X}$ is a positive (negative)-definite matrix. If $\boldsymbol{X} > 0$, then $\boldsymbol{X}^{1/2}$ denotes the positive-definite square root matrix of $\boldsymbol{X}$ such that $\boldsymbol{X}^{(1/2)\mathrm{T}}\boldsymbol{X}^{1/2} = \boldsymbol{X}$. $\lambda_{\max}(\boldsymbol{X})$ denotes the maximum eigenvalue of the matrix $\boldsymbol{X}$. $\|\cdot\|$ denotes the Euclidean norm or a matrix norm induced by the Euclidean norm. $|\cdot|$ denotes modulus of a complex number. $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform, and the inverse Laplace transform, respectively. $*$ denotes convolution.

2 Problem formulation

A class of linear state-delayed systems is given as follows

$$
\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_0\boldsymbol{x}(t) + \boldsymbol{A}_1\boldsymbol{x}(t - \tau) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{v}(t) \\
\boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t)
\end{aligned}
\tag{1}
$$

whose initial condition is

$$
\boldsymbol{x}(\theta) = \boldsymbol{\phi}(\theta), \quad \theta \in [-\tau, 0]
$$

where $\boldsymbol{A}_0, \boldsymbol{A}_1 \in \mathbb{R}^{n \times n}$, $\boldsymbol{B} \in \mathbb{R}^{n \times m}$, $\boldsymbol{C} \in \mathbb{R}^{m \times n}$, $\boldsymbol{x}(t)$, $\boldsymbol{v}(t) \in \mathbb{R}^{n \times 1}$, $\boldsymbol{y}(t)$, $\boldsymbol{u}(t) \in \mathbb{R}^{m \times 1}$, t, $\tau \in \mathbb{R}$ with $t \geq 0$, $\tau > 0$. $\boldsymbol{x}(t)$ is the state of the system, τ is the time delay, $\boldsymbol{y}(t)$ is the output of the system. $\boldsymbol{v}(t)$ denotes the unknown periodic disturbance, that is, $\boldsymbol{v}(t) = \boldsymbol{v}(t + T)$. $\boldsymbol{\phi}(t)$ is a bounded vector valued function representing the initial condition function. The control objective is to track a desired trajectory $\boldsymbol{y}_{\mathrm{d}}(t)$ with period T. In this paper, we assume that the state of the system on $[-\infty, -\tau]$ is zero.

On system (1), we impose the following assumption:

Assumption 1: $\boldsymbol{A}_1$ and τ are unknown, but $\boldsymbol{A}_1\boldsymbol{A}_1^{\mathrm{T}} < \gamma\boldsymbol{I}_n$ and $\gamma > 0$ is known. Matrices $\boldsymbol{A}_0$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are known, and the pair $\{\boldsymbol{A}_0, \boldsymbol{B}\}$ is controllable.

The controller $\boldsymbol{u}(t)$ in (1) is designed as

$$
\boldsymbol{u}(t) = \boldsymbol{u}^{\mathrm{f}}(t) + \boldsymbol{u}^{\mathrm{b}}(t)
$$

where $\boldsymbol{u}^{\mathrm{f}}$ and $\boldsymbol{u}^{\mathrm{b}}$ denote the learning-based feedforward [6] control signal and the feedback control signal, respectively. Suppose that $\boldsymbol{A}_0$ is a stable matrix. Otherwise, a controller $\boldsymbol{u}^{\mathrm{b}}(t)$ can be designed to stabilise it since the pair $\{\boldsymbol{A}_0, \boldsymbol{B}\}$ is controllable by Assumption 1. Without loss of generality, we assume $\boldsymbol{A}_0$ is a stable matrix and let $\boldsymbol{u}^{\mathrm{b}}(t) \equiv 0$ here for simplicity. In the following Section, we will restrict our attention to the design of $\boldsymbol{u}^{\mathrm{f}}$.

3 Repetitive control problem

In Section 3.1, a repetitive controller is proposed to track periodic reference signals for a class of linear state-delayed systems subject to uncertainties and periodic disturbances. In order to relax the stability condition on the closed-loop system, a modified repetitive controller is developed in Section 3.2.

3.1 Repetitive controller

In this section, we first make

Assumption 2: There exists a bounded and continuous control $\boldsymbol{u}_{\mathrm{d}}(t) = \boldsymbol{u}_{\mathrm{d}}(t + T)$ which, when substituted for $\boldsymbol{u}(t)$ in (1), causes $\boldsymbol{y}(t)$ to track $\boldsymbol{y}_{\mathrm{d}}(t)$ perfectly. Namely, we have the reference system

$$
\begin{aligned}
\dot{\boldsymbol{x}}_{\mathrm{d}}(t) &= \boldsymbol{A}_0\boldsymbol{x}_{\mathrm{d}}(t) + \boldsymbol{A}_1\boldsymbol{x}_{\mathrm{d}}(t - \tau) + \boldsymbol{B}\boldsymbol{u}_{\mathrm{d}}(t) + \boldsymbol{v}(t) \\
\boldsymbol{y}_{\mathrm{d}}(t) &= \boldsymbol{C}\boldsymbol{x}_{\mathrm{d}}(t)
\end{aligned}
\tag{2}
$$

Here $\boldsymbol{x}_{\mathrm{d}}(t) \in \mathbb{R}^{n \times 1}$ plays the role of 'desired state'.

Remark 1: Taking the Laplace transform of (2), we obtain

$$
\boldsymbol{y}_{\mathrm{d}}(s) = \boldsymbol{G}(s)\boldsymbol{u}_{\mathrm{d}}(s) + \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A}_0 - \boldsymbol{A}_1 e^{-\tau s})^{-1}\boldsymbol{v}(s)
$$

where $\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A}_0 - \boldsymbol{A}_1 e^{-\tau s})^{-1}\boldsymbol{B} \in \mathbb{R}^{m \times m}$. If $\boldsymbol{G}(s)$ is invertible, then $\boldsymbol{u}_{\mathrm{d}}(s)$ can be written as

$$
\boldsymbol{u}_{\mathrm{d}}(s) = \boldsymbol{G}(s)^{-1}\boldsymbol{y}_{\mathrm{d}}(s) - \boldsymbol{G}(s)^{-1}\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A}_0 - \boldsymbol{A}_1 e^{-\tau s})^{-1}\boldsymbol{v}(s)
\tag{3}
$$

There exist desired trajectories, such as triangular and square waves, which cannot be tracked perfectly by the system (2) no matter what $\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{B}, \boldsymbol{C}$ are. Therefore we need to impose an appropriate differentiability condition on the desired trajectory $\boldsymbol{y}_{\mathrm{d}}(t)$ and the disturbance $\boldsymbol{v}(t)$. For example

suppose $G(s) = 1/(1 - e^{-\tau s} + s)$ and $\boldsymbol{v}(s) \equiv 0$; thus $\boldsymbol{u}_d(s) = (1 - e^{-\tau s} + s)\boldsymbol{y}_d(s)$, which implies $\boldsymbol{u}_d(t) = \boldsymbol{y}_d(t) - \boldsymbol{y}_d(t - \tau) + \dot{\boldsymbol{y}}_d(t)$; the appropriate differentiability condition means that $\boldsymbol{y}_d(t)$ is first-order differentiable. For any periodic, bounded and continuous trajectory and disturbance which satisfy the appropriate differentiability condition, if $G(s)$ is invertible and $G(s)^{-1}$, $G(s)^{-1}C(sI - A_0 - A_1 e^{-\tau s})^{-1}$ are stable transfer matrices, then $\boldsymbol{u}_d(t) = \mathcal{L}^{-1}[\boldsymbol{u}_d(s)]$ in (3) is periodic, bounded, continuous and unique [7] (since we restrict our attention to functions that are continuous on $[0, +\infty)$). In fact, Assumption 2 may also hold in other cases.

By subtracting (1) from (2), the error dynamic system is described by

$$\delta\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_0\delta\boldsymbol{x}(t) + \boldsymbol{A}_1\delta\boldsymbol{x}(t - \tau) + \boldsymbol{B}\delta\boldsymbol{u}(t)$$
$$\delta\boldsymbol{y}(t) = \boldsymbol{C}\delta\boldsymbol{x}(t) \tag{4}$$

where $\delta\boldsymbol{x} \triangleq \boldsymbol{x}_d - \boldsymbol{x}$, $\delta\boldsymbol{y} \triangleq \boldsymbol{y}_d - \boldsymbol{y}$ and $\delta\boldsymbol{u} \triangleq \boldsymbol{u}_d - \boldsymbol{u}^f$.

Suppose the repetitive controller is designed as follows

$$\boldsymbol{u}^f(t) = \boldsymbol{u}^f(t - T) + \boldsymbol{K}\delta\boldsymbol{y}(t) \tag{5}$$

where $\boldsymbol{K} \in \mathbb{R}^{m \times m}$.

By utilising the property of $\boldsymbol{u}_d(t)$ by Assumption 2, that is, $\boldsymbol{u}_d(t) = \boldsymbol{u}_d(t + T)$, (5) can be written as

$$\delta\boldsymbol{u}(t) = \delta\boldsymbol{u}(t - T) - \boldsymbol{K}\boldsymbol{C}\delta\boldsymbol{x}(t) \tag{6}$$

Combining (4) and (6), we can obtain a neutral delay equation as follows

$$\delta\dot{\boldsymbol{x}}(t) - \boldsymbol{H}\delta\dot{\boldsymbol{x}}(t - T) = (\boldsymbol{A}_0 - \boldsymbol{B}\boldsymbol{K}\boldsymbol{C})\delta\boldsymbol{x}(t) - \boldsymbol{A}_0\delta\boldsymbol{x}(t - T)$$
$$+ \boldsymbol{A}_1\delta\boldsymbol{x}(t - \tau) - \boldsymbol{A}_1\delta\boldsymbol{x}(t - \tau - T) \tag{7}$$

where $\boldsymbol{H} = \boldsymbol{I}_n$.

For neutral delay equations as (7), the assumption that all the eigenvalues of $\boldsymbol{H}$ are inside the unit circle is required in much existing literature, such as [8, 9]. However, the assumption is not satisfied by the system (7), where $\boldsymbol{H} = \boldsymbol{I}_n$. In fact, (7) is a critical case of the neutral delay equation [10] and its characteristic equation has an infinite sequence of roots whose real parts approach zero [11]. The stability of such a system is difficult to analyse.

In this paper, for some special systems, a sufficient stability condition of (7) can be derived in terms of LMIs.

Lemma 1: Consider system (1) with Assumptions 1–2 under the following control law

$$\boldsymbol{u}^f(t) = \boldsymbol{u}^f(t - T) + \boldsymbol{K}_0(t)\boldsymbol{B}^T\boldsymbol{P}\delta\boldsymbol{x}(t)$$
$$\boldsymbol{u}^f(t) = 0, \ t \in [-T, 0),$$
$$\boldsymbol{K}_0(t) = \begin{cases} 0 & t \in [-T, 0) \\ \boldsymbol{K}_1(t) & t \in [0, T) \\ \boldsymbol{K} & t \in [T, +\infty) \end{cases} \tag{8}$$

where every element of the matrix $\boldsymbol{K}_0(t) \in \mathbb{R}^{m \times m}$ is continuous on $[-T, +\infty)$ and $\boldsymbol{K} > 0$. If there exist $0 < \boldsymbol{P} = \boldsymbol{P}^T \in \mathbb{R}^{n \times n}$ and $0 < \alpha \in \mathbb{R}$ such that

$$\begin{bmatrix} \boldsymbol{P}\boldsymbol{A}_0 + \boldsymbol{A}_0^T\boldsymbol{P} + \alpha\boldsymbol{I}_n & \boldsymbol{P} \\ \boldsymbol{P} & -\gamma^{-1}\alpha\boldsymbol{I}_n \end{bmatrix} < 0 \tag{9}$$

then

$$\lim_{t \to +\infty} \int_{t-T}^t \|\delta\boldsymbol{y}(\theta)\|^2 d\theta = 0$$

Proof: Choose the Lyapunov–Krasovskii functional to be

$$V_1(t, \delta\boldsymbol{x}) = \delta\boldsymbol{x}^T(t)\boldsymbol{P}\delta\boldsymbol{x}(t) + \alpha\int_{t-\tau}^t \delta\boldsymbol{x}^T(s)\delta\boldsymbol{x}(s) ds$$

Taking the derivative of $V_1(t, \delta\boldsymbol{x})$ along the solution of (4) yields

$$\dot{V}_1(t, \delta\boldsymbol{x}) = \delta\boldsymbol{x}^T(t)(\boldsymbol{P}\boldsymbol{A}_0 + \boldsymbol{A}_0^T\boldsymbol{P} + \alpha\boldsymbol{I}_n)\delta\boldsymbol{x}(t)$$
$$+ 2\delta\boldsymbol{x}^T(t)\boldsymbol{P}\boldsymbol{A}_1\delta\boldsymbol{x}(t - \tau)$$
$$- \alpha\delta\boldsymbol{x}^T(t - \tau)\delta\boldsymbol{x}(t - \tau) + 2\delta\boldsymbol{x}^T(t)\boldsymbol{P}\boldsymbol{B}\delta\boldsymbol{u}(t)$$
$$\leq \delta\boldsymbol{x}^T(t)\boldsymbol{R}_1\delta\boldsymbol{x}(t) + 2\delta\boldsymbol{x}^T(t)\boldsymbol{P}\boldsymbol{B}\delta\boldsymbol{u}(t)$$

where $\boldsymbol{R}_1 = \boldsymbol{P}\boldsymbol{A}_0 + \boldsymbol{A}_0^T\boldsymbol{P} + \alpha\boldsymbol{I}_n + \alpha^{-1}\boldsymbol{P}\boldsymbol{A}_1\boldsymbol{A}_1^T\boldsymbol{P}$.

Using Assumption 1, we can obtain, $\boldsymbol{P}\boldsymbol{A}_1\boldsymbol{A}_1^T\boldsymbol{P} \leq \gamma\boldsymbol{P}\boldsymbol{P}$, thus

$$\dot{V}_1(t, \delta\boldsymbol{x}) \leq \delta\boldsymbol{x}^T(t)\boldsymbol{R}_2\delta\boldsymbol{x}(t) + 2\delta\boldsymbol{x}^T(t)\boldsymbol{P}\boldsymbol{B}\delta\boldsymbol{u}(t) \tag{10}$$

where $\boldsymbol{R}_2 = \boldsymbol{P}\boldsymbol{A}_0 + \boldsymbol{A}_0^T\boldsymbol{P} + \alpha\boldsymbol{I}_n + \alpha^{-1}\gamma\boldsymbol{P}\boldsymbol{P}$.

Since $\boldsymbol{u}_d(t) = \boldsymbol{u}_d(t - T)$ by Assumption 2, (8) becomes

$$\delta\boldsymbol{u}(t) = \delta\boldsymbol{u}(t - T) - \boldsymbol{K}_0(t)\boldsymbol{B}^T\boldsymbol{P}\delta\boldsymbol{x}(t) \tag{11}$$

Using (11), we have the following equation

$$\delta\boldsymbol{u}^T(t)\boldsymbol{K}^{-1}\delta\boldsymbol{u}(t) - \delta\boldsymbol{u}^T(t - T)\boldsymbol{K}^{-1}\delta\boldsymbol{u}(t - T)$$
$$= -\left[\boldsymbol{K}_0(t)\boldsymbol{B}^T\boldsymbol{P}\delta\boldsymbol{x}(t)\right]^T\boldsymbol{K}^{-1}\left[\boldsymbol{K}_0(t)\boldsymbol{B}^T\boldsymbol{P}\delta\boldsymbol{x}(t)\right]$$
$$- 2\delta\boldsymbol{x}^T(t)\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}_0(t)\boldsymbol{K}^{-1}\delta\boldsymbol{u}(t) \tag{12}$$

Define a non-negative function $W(t, \delta x, \delta u) \in \mathbb{R}$ as follows

$$W(t, \delta x, \delta u) = V_1(t, \delta x) + \int_{t-T}^{t} \delta u^{\mathrm{T}}(\tau) K^{-1} \delta u(\tau)\, d\tau \quad (13)$$

Taking the derivative of (7), we obtain the following expression

$$\dot{W}(t, \delta x, \delta u) = \dot{V}_1(t, \delta x) + [\delta u^{\mathrm{T}}(t) K^{-1} \delta u(t)$$
$$- \delta u^{\mathrm{T}}(t-T) K^{-1} \delta u(t-T)] \quad (14)$$

Applying (10) and (12) to (14) yields

$$\dot{W}(t, \delta x, \delta u) \le \delta x^{\mathrm{T}}(t) R_2 \delta x(t) + [I_m - K_0(t) K^{-1}]$$
$$\times \delta x^{\mathrm{T}}(t) PB \delta u(t)$$

Since $u^{\mathrm{f}}(t) = K_1(t) B^{\mathrm{T}} P \delta x(t)$, $t \in [0, T)$, $\delta x(t)$ is bounded when $t \in [0, T)$. For $t \in [T, +\infty)$, we have $\dot{W}(t, \delta x, \delta u) \le \delta x^{\mathrm{T}}(t) R_2 \delta x(t)$.

Suppose there exist $0 < P = P^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ and $0 < \alpha \in \mathbb{R}$ such as $R_2 < 0$. Then, $\|\delta x(t)\|$ is bounded; moreover, by utilising Barbalat's Lemma as in [13], the equation below holds

$$\lim_{t \to +\infty} \int_{t-T}^{t} \|\delta x(\theta)\|^2 d\theta = 0 \quad \text{and} \quad \lim_{t \to +\infty} \int_{t-T}^{t} \|\delta y(\theta)\|^2 d\theta = 0$$

Using Schur Complement [12], the following inequalities are equivalent to each other.

$$R_2 < 0 \Leftrightarrow (9)$$

Therefore this concludes the proof. □

The knowledge of $x_{\mathrm{d}}(t)$ usually means that the reference model is known a priori. However, one may sometimes not have access to $x_{\mathrm{d}}(t)$ and/or $x(t)$. In this case, a controller that requires only the output error is given.

Theorem 1: Consider system (1) with Assumptions 1–2 under the following control law

$$u^{\mathrm{f}}(t) = u^{\mathrm{f}}(t-T) + K_0(t) \delta y(t)$$
$$u^{\mathrm{f}}(t) = 0,\ t \in [-T, 0) \quad (15)$$

If there exist $0 < P = P^{\mathrm{T}} \in \mathbb{R}^{n \times n}$ and $0 < \alpha \in \mathbb{R}$ which satisfy (9) and $B^{\mathrm{T}} P = C$, then

$$\lim_{t \to +\infty} \int_{t-T}^{t} \|\delta y(\theta)\|^2 d\theta = 0$$

Proof: By using Lemma 1, the proof is trivial. □

Remark 2: The controller (15) differs from (5) only on the interval $[0, T)$. As [13], the reason for introducing $K_0(t)$ is to

ensure that $u^{\mathrm{f}}(t)$ is continuous when $t \in [-T, +\infty)$. We have the fact that $u^{\mathrm{f}}(0^-) = 0$ and $u^{\mathrm{f}}(0^+) = B^{\mathrm{T}} P \delta x(0)$. $\delta x(0)$, the initial state error, is usually unequal to zero. Therefore, $u^{\mathrm{f}}(t)$ is discontinuous at $t = 0$ with $K_0(t) \equiv K$ and will make the actuator switch and excite high-frequency modes of the system. Whereas $u^{\mathrm{f}}(0^+) = 0$ holds when $K_0(t)$ is defined as in (8). It implies that $u^{\mathrm{f}}(t)$ is continuous when $t \in [-T, T)$. This is easy to prove that $u^{\mathrm{f}}(t)$ is continuous when $t \in [-T, +\infty)$ by induction.

Remark 3: A necessary condition for Theorem 1 to hold is $CB \ne 0$. Otherwise, $B^{\mathrm{T}} PB = CB = 0$, which contradicts the fact that P is a positive-definite matrix. In [14], when only the output error signal is available for feedback, a repetitive controller is proposed for the positive real system as $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, where $CB \ne 0$ is also the necessary condition.

3.2 Modified repetitive controller

As shown in Section 3.1, asymptomatic tracking can be achieved by the proposed repetitive controller. However, the condition of Theorem 1 is difficult to satisfy with the restriction $B^{\mathrm{T}} P = C$. Therefore a modified repetitive controller is developed to track periodic reference signals subject to periodic disturbances for a class of linear state-delayed systems. Since the transfer function of a time-delay system is not rational, the theory on the modified RC systems proposed by Hara *et al.* [5] cannot be applied directly. Therefore the first task is to extend the theory on the modified RC system so that it can apply to systems represented by irrational transfer functions.

In Fig. 1, $G_v(s) = C(sI_n - A_0 - A_1 e^{-s\tau})^{-1} \in \mathbb{R}^{m \times n}$.

The minimal realisation of the low-pass filter $q(s)$ is assumed as follows

$$\dot{x}_p(t) = A_p x_p(t) + B_p u_p(t)$$
$$y_p(t) = C_p x_p(t)$$

where $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times 1}$, $C_p \in \mathbb{R}^{1 \times n_p}$, $y_p(t)$, $u_p(t) \in \mathbb{R}$ and $x_p(t) \in \mathbb{R}^{n_p \times 1}$ is the state variable.
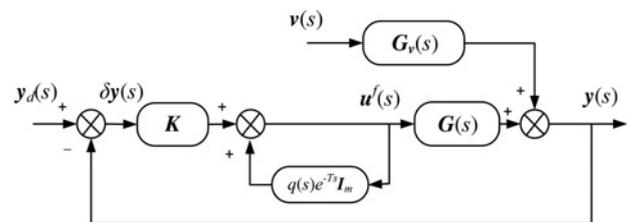
Figure 1 *Modified repetitive control system*

The structure of the modified repetitive controller in Fig. 1 can be expressed as follows

$$\dot{x}_p(t) = A_p x_p(t) + B_p C_p x_p(t - T)$$
$$- B_p KC x(t) + B_p K y_d(t) \qquad (16)$$
$$u^f(t) = C_p x_p(t - T) - KC x(t) + K y_d(t)$$

Combining (1) and (16), we have

$$\dot{Z}(t) = D_1 Z(t) + D_2 Z(t - \tau) + D_3 Z(t - T) + B_D \xi(t)$$
$$\delta y(t) = C_D Z(t) + D_D \xi(t) \qquad (17)$$

where

$$Z = \begin{bmatrix} x \\ x_p \end{bmatrix}, \quad D_1 = \begin{bmatrix} A_0 - BKC & 0 \\ -B_p KC & A_p \end{bmatrix},$$

$$D_2 = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 0 & BC_p \\ 0 & B_p C_p \end{bmatrix}$$

$$B_D = \begin{bmatrix} BK & I_n \\ B_p K & 0 \end{bmatrix}, \quad C_D = \begin{bmatrix} -C & 0 \end{bmatrix},$$

$$D_D = \begin{bmatrix} I_m & 0 \end{bmatrix}, \quad \xi = \begin{bmatrix} y_d \\ v \end{bmatrix}$$

Remark 4: $G_{er}(s) \in \mathbb{R}^{m \times (m+n)}$ denotes the transfer function matrix from $\xi(t)$ to $\delta y(t)$ and can be written as

$$G_{er}(s) = C_D(sI_{n+n_p} - D_1 - D_2 e^{-s\tau} - D_3 e^{-sT})^{-1} B_D + D_D$$

If the zero solutions of (18), namely of

$$\dot{Z}(t) = D_1 Z(t) + D_2 Z(t - \tau) + D_3 Z(t - T) \qquad (18)$$

are exponentially stable, then $G_{er}(s)$ is exponentially stable. In fact, the uniform asymptotic stability property is equivalent to the exponential stability property for the system [11, 15]. This implies that if the system in Fig. 1 is internally stable [16], then $G_{er}(s)$ is exponentially stable.

Before proceeding further with the development of this work, the following theorem is needed. For simplicity, $\mathbb{R}_s$ denotes the class of transfer functions of stable linear time-delay systems. We introduce a sequence $\{q_i(s)|i = 1, 2, \dots\}$ which will represent a family of filters and $\bar{q}(s) = \lim_{i \to +\infty} q_i(s)$. $\Phi_i(t)$ represents the fundamental solution of (18) with different low-pass filters $q(s) = q_i(s)$. Theorem 2 in this section is an extension of Theorem 2 in [5].

Theorem 2: Suppose (i) for an arbitrary but fixed bounded interval $[-\omega_f, \omega_f]$, $\bar{q}(s) = 1$ holds on $[-\omega_f, \omega_f]$ and reference signal $\xi(t)$ with period T contains only frequencies lower than ω_f; (ii) $\|\Phi_i(t)\| \le Ke^{-\alpha t}$ independently of i, where K, α are positive real numbers; (iii) $G_v(s) \in \mathbb{R}_s$. Then tracking error

$\delta y_i(t)$ in the modified RC system (17) with $q = q_i$ satisfies

$$\lim_{i \to +\infty} \limsup_{k \to +\infty} \|\delta y_i(t)\|_{[kT,(k+1)T]} = 0 \qquad (19)$$

where $\| \cdot \|_{[kT,(k+1)T]}$ denotes the L_∞-norm on $[kT, (k+1)T]$.

Proof: In Fig. 1, $G_{er}^i(s)$ denotes the transfer function matrix from $\xi(s)$ to $\delta y_i(s)$, where $\xi(s)$ and $\delta y_i(s)$ are the Laplace transforms of $\xi(t)$ and $\delta y_i(t)$, respectively. $\delta y_i(s)$ can be written as follows

$$\delta y_i(s) = \delta y_{i,1}(s) + \delta y_{i,2}(s)$$

where $\delta y_{i,1}(s) = G_{er,1}^i(s) y_d(s)$, $\delta y_{i,2}(s) = G_{er,2}^i(s) v(s)$, $G_{er,1}^i(s) \in \mathbb{R}^{m \times m}$ and $G_{er,2}^i(s) \in \mathbb{R}^{m \times n}$.

Since $G_v(s) \in \mathbb{R}_s$ and $\|\Phi_i(t)\| \le Ke^{-\alpha t}$ holds independently of i, $G_{er,1}^i(s)$ is exponentially stable independently of i, consequently, $G_{er,2}^i(s) = -G_{er,1}^i(s)G_v(s)$ is also exponentially stable independently of i.

For simplicity, we only consider $G_{er,1}^i(s)$, which is written as follows

$$G_{er,1}^i(s) = [I_m + (I_m - q_i(s)e^{-Ts} I_m)^{-1} G(s)]^{-1}$$
$$= [1 - q_i(s)e^{-Ts}][I_m - q_i(s)e^{-Ts} I_m + G(s)]^{-1}$$

As seen above, every element of $G_{er,1}^i(s)$ possesses zeros at $\{s | 1 - q_i(s)e^{-Ts} = 0\}$. Denote $G_{p_1,p_2}^i(s)$ is the element in the p_1th row and p_2th column of $G_{er,1}^i(s)$, $1 \le p_1, p_2 \le m$. Then $G_{p_1,p_2}^i(s)$ is also exponentially stable independently of i and possesses zeros at $\{s | 1 - q_i(s)e^{-Ts} = 0\}$.

For $r(t) \in \mathbb{R}$ with period T contains only frequencies lower than ω_f, suppose the following equation holds

$$\lim_{i \to +\infty} \lim_{k \to +\infty} \|\delta y_{p_1,p_2}^i(t)\|_{[kT,(k+1)T]} = 0 \qquad (20)$$

where $\delta y_{p_1,p_2}^i(t) = \mathcal{L}^{-1}[G_{p_1,p_2}^i(s)] * r(t)$ with $q = q_i$. Then (19) is satisfied because every element of $\delta y_i(t)$ is a linear combination of $\delta y_{p_1,p_2}^i(t)$, $1 \le p_1, p_2 \le m$. Therefore, in the remainder of the proof, we only need to consider whether (20) holds.

Let N be the largest integer such that $|\omega_N| < \omega_f$, where $\omega_k = 2\pi kj/T$. Given any $0 < \epsilon < 1$, there exist zeros $\alpha_k^i \pm j\beta_k^i$, $k = 0, 1, \dots, N(\beta_0 = 0, \beta_{-k}^i = -\beta_k^i)$ of $G_{p_1,p_2}^i(s)$ such that

$$|\alpha_k^i + j\beta_k^i - j\omega_k| < \epsilon, \ k = 0, \ \pm 1, \ \dots, \ \pm N \qquad (21)$$

for all sufficiently large i.

Since $G^i_{p_1,p_2}(\alpha^i_0) = 0$ and $G^i_{p_1,p_2}(\alpha^i_k + j\beta^i_k) = 0$, $G^i_{p_1,p_2}(s)$ can be written as

$$G^i_{p_1,p_2}(s) = \tilde{G}^i_{p_1,p_2,0}(s)(s - \alpha^i_0)$$

$$G^i_{p_1,p_2}(s) = \tilde{G}^i_{p_1,p_2,k}(s)[(s - \alpha^i_k)^2 + (\beta^i_k)^2], \quad k = 1, \ldots, N$$

Since $r(t)$ contains only frequencies lower than ω_f, it can be written as

$$r(t) = \sum_{k=0}^{N} [a_k \sin(\omega_k t) + b_k \cos(\omega_k t)]$$

Then $\delta y^i_{p_1,p_2}(t)$ in (20) can be written as follows

$$\delta y^i_{p_1,p_2}(t) = \sum_{k=0}^{N} a_k \mathcal{L}^{-1}[G^i_{p_1,p_2}(s) \cdot \omega_k/(s^2 + \omega_k^2)]$$
$$+ \sum_{k=0}^{N} b_k \mathcal{L}^{-1}[G^i_{p_1,p_2}(s) \cdot s/(s^2 + \omega_k^2)]$$
$$= \sum_{k=0}^{N} a_k \mathcal{L}^{-1}\{\tilde{G}^i_{p_1,p_2,k}(s) \cdot \omega_k[(s - \alpha^i_k)^2$$
$$+ (\beta^i_k)^2]/(s^2 + \omega_k^2)\}$$
$$+ \sum_{k=0}^{N} b_k \mathcal{L}^{-1}\{s\tilde{G}^i_{p_1,p_2,k}(s) \cdot [(s - \alpha^i_k)^2$$
$$+ (\beta^i_k)^2]/(s^2 + \omega_k^2)\} \tag{22}$$

Without loss of generality, we consider $r(t) = \sin(\omega_k t)$. In this case, $\delta y^i_{p_1,p_2}(t)$ can be represented as follows

$$\delta y^i_{p_1,p_2}(t) = \omega_k \mathcal{L}^{-1}[\tilde{G}^i_{p_1,p_2,k}(s)] * \delta(t)$$
$$+ \mathcal{L}^{-1}[\tilde{G}^i_{p_1,p_2,k}(s)] * \tilde{u}(t) \tag{23}$$

where $\delta(t)$ denotes the Dirac delta function, $\tilde{u}(t) = [(\alpha^i_k)^2 + (\beta^i_k)^2 - \omega_k^2] \sin(\omega_k t) - 2\alpha^i_k \omega_k \cos(\omega_k t)$.

Since $G^i_{p_1,p_2}(s)$ is exponentially stable independently of i, $\tilde{G}^i_{p_1,p_2,k}(s)$ is also exponentially stable independently of i. Then (23) is further written as follows [5]

$$|\delta y^i_{p_1,p_2}(t)| \leq C_0 e^{-\bar{\alpha}t} + C_1 \sup_{t \in [0,+\infty)} |\tilde{u}(t)|$$
$$\leq C_0 e^{-\bar{\alpha}t} + C_2 \epsilon \tag{24}$$

where ϵ is defined in (21) and $\bar{\alpha}, C_0, C_1, C_2$ are positive numbers independent of i. It is easy to verify that $\delta y^i_{p_1,p_2}(t)$ in (22) also has the form as in (24).

By (21), ϵ can be made arbitrarily small when i is sufficiently large. It follows that (20) holds. This concludes the proof. $\qquad\square$

Remark 5: The low-pass filter $q(s)$, known as the 'q-filter' [5], should be appropriately selected for a trade-off between good tracking performance and stability margin. When the bandwidth of $q(s)$ increases, the tracking performance also improves but the stability margin decreases, and vice versa [17]. In particular, if $G_{er}(s)$ is stable with $q(s) = 1$ and $G_v(s) \in \mathbb{R}_s$, then asymptomatic tracking performance can be achieved. This in fact has been discussed in Theorem 1.

Based on the conclusion of Theorem 2, we only need to consider the internal stability of the system shown in Fig. 1, that is, the stability of (18) and (25). (The stability of (25) gives $G_v(s) \in \mathbb{R}_s$)

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) \tag{25}$$

The stability of (25) can be determined by (9). For (18) and (25), note that the uniform asymptotic stability is equivalent to the exponential stability. Then, up to now, the tracking problem has been converted from the uniform asymptotic stability of a neutral delay system (7) to the uniform asymptotic stability of the more familiar time-delay systems (18) and (25). Generally speaking, the latter is easier to handle. The following theorem gives a sufficient condition for the uniform asymptotic stability of system (18) using the frequency-domain approach.

Theorem 3: Consider the system (18) with Assumption 1. If (i) D_1 has no eigenvalues in the closed left half-plane, (ii) $\lambda_{\max}[M_1(j\omega)] < 0.5$, $\omega \in \mathbb{R}$, then $Z(t) = 0$ is uniformly asymptotically stable in (18), where

$$M_1(j\omega) = (j\omega I_{n+n_p} - D_1)^{-1}(D_3 D_3^T + \gamma I_{n+n_p})$$
$$\times (-j\omega I_{n+n_p} - D_1^T)^{-1}$$

Proof: Let $Z_1(t)$, $Z_2(t)$, $Z_3(t)$ in (18) be redefined as $Z_1(t) = Z(t - T)$, $Z_2(t) = Z(t - \tau)$, $Z_3(t) = Z(t)$, respectively. Then (18) can be rewritten as

$$\begin{pmatrix} Z_1(t + T) \\ Z_2(t + \tau) \\ \dot{Z}_3(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & I_{n+n_p} \\ 0 & 0 & I_{n+n_p} \\ D_3 & D_2 & D_1 \end{pmatrix} \begin{pmatrix} Z_1(t) \\ Z_2(t) \\ Z_3(t) \end{pmatrix}$$

By using Lemma 4 in [18], we conclude that if and only if D_1 has no eigenvalues in the closed left half-plane, that is, condition (i); and $\det[I_{2(n+n_p)} - z M_2(j\omega)] \neq 0$ in $\{z \in \mathbb{C}^1 | |z| \leq 1\}$ for $\omega \in \mathbb{R}$ hold, then $Z(t) = 0$ is uniformly asymptotically stable in (18), where

$$M_2(s) = \begin{pmatrix} (sI_{n+n_p} - D_1)^{-1}D_3 & (sI_{n+n_p} - D_1)^{-1}D_2 \\ (sI_{n+n_p} - D_1)^{-1}D_3 & (sI_{n+n_p} - D_1)^{-1}D_2 \end{pmatrix}$$

The condition $\lambda_{\max}[M_2(j\omega)M_2^T(-j\omega)] < 1$ implies that

$\det\left[I_{2(n+n_p)} - zM_2(j\omega)\right] \neq 0$ in $\{z \in \mathbb{C}^1 \| |z| \leq 1\}$ for $\omega \in \mathbb{R}$. Therefore the remainder of the proof focuses on proving

$$\lambda_{\max}[M_2(j\omega)M_2^T(-j\omega)] < 1$$

$M_2(j\omega)M_2^T(-j\omega)$ can be written as

$$M_2(j\omega)M_2^T(-j\omega) = \begin{pmatrix} M_3(j\omega) & M_3(j\omega) \\ M_3(j\omega) & M_3(j\omega) \end{pmatrix}$$

where $M_3(j\omega) = (j\omega I_{n+n_p} - D_1)^{-1} (D_3 D_3^T + D_2 D_2^T) (-j\omega I_{n+n_p} - D_1^T)^{-1}$.

Using Assumption 1, we obtain

$$D_2 D_2^T = \begin{bmatrix} A_1 A_1^T & 0 \\ 0 & 0 \end{bmatrix} \leq \gamma I_{n+n_p}$$

thus

$$M_2(j\omega)M_2^T(-j\omega) \leq \begin{pmatrix} M_1(j\omega) & M_1(j\omega) \\ M_1(j\omega) & M_1(j\omega) \end{pmatrix} \qquad (26)$$

Since

$$\begin{pmatrix} M_1(j\omega) & M_1(j\omega) \\ M_1(j\omega) & M_1(j\omega) \end{pmatrix} = N^{-1} \begin{pmatrix} 2M_1(j\omega) & M_1(j\omega) \\ 0 & 0 \end{pmatrix} N$$

$$N = \begin{pmatrix} I_{n+n_p} & 0 \\ -I_{n+n_p} & I_{n+n_p} \end{pmatrix}$$

hence $\lambda_{\max}[M_2(j\omega)M_2^T(-j\omega)] \leq 2\lambda_{\max}[M_1(j\omega)]$ by (26).

Therefore if condition (ii) holds, then $\lambda_{\max}[M_2(j\omega)M_2^T(-j\omega)] < 1$ which concludes the proof. □

The stability of (25) can also be determined by Theorem 3 as it is only a special case of system (18). Although the stability of (18) can be determined by its characteristic function and the criterion may be less conservative, the stability criterion will become more and more difficult to verify as the system dimension increases. In the following section, the delay-independent criterion of system (18) is derived in terms of LMIs. This makes the criterion quite feasible with the aid of a computer.

Theorem 4: Consider the system (18) with Assumption 1 under the control law (16). If there exist $0 < Q_i = Q_i^T \in \mathbb{R}^{(n+n_p) \times (n+n_p)}$, $i = 1, 2$, $0 < \alpha \in \mathbb{R}$ and a matrix

$K \in \mathbb{R}^{m \times m}$ such that

$$\begin{bmatrix} Q_1 D_1 + D_1^T Q_1 + \alpha I_{n+n_p} + Q_2 & Q_1 & Q_1 D_3 \\ Q_1 & -\gamma^{-1}\alpha I_{n+n_p} & 0 \\ D_3^T Q_1 & 0 & -Q_2 \end{bmatrix}$$
$$< 0 \qquad (27)$$

then $Z(t) = 0$ is uniformly asymptotically stable in (18).

Proof: Choose the Lyapunov–Krasovskii functional to be

$$V_2(t, Z) = Z^T(t)Q_1 Z(t) + \alpha \int_{t-\tau}^{t} Z^T(s)Z(s)\,ds$$
$$+ \int_{t-T}^{t} Z^T(s)Q_2 Z(s)\,ds$$

Taking the derivative of $V_2(t, Z)$ along the solution of (18) yields

$$\dot{V}_2(t, Z) = Z^T(t)R_3 Z(t) - H_1^T H_1 - H_2^T H_2$$
$$\leq Z^T(t)R_3 Z(t) \qquad (28)$$

where

$$R_3 = D_1^T Q_1 + Q_1 D_1 + \alpha I + Q_2 + \alpha^{-1} Q_1 D_2 D_2^T Q_1$$
$$+ Q_1 D_3 Q_2^{-1} D_3^T Q_1$$

$$H_1 = \left(\sqrt{\alpha}\right)^{-1} D_2^T Q_1 Z(t) - \sqrt{\alpha} Z(t-\tau)$$

$$H_2 = \left(Q_2^{\frac{1}{2}}\right)^{-1} D_3^T Q_1 Z(t) - Q_2^{\frac{1}{2}} Z(t-T)$$

Using Assumption 1, we can obtain $D_2 D_2^T \leq \gamma I_{n+n_p}$. Thus (28) becomes

$$\dot{V}_2(Z, t) \leq Z^T(t)R_4 Z(t)$$

where $R_4 = D_1^T Q_1 + Q_1 D_1 + \alpha I + Q_2 + \gamma \alpha^{-1} Q_1 Q_1 + Q_1 D_3 Q_2^{-1} D_3^T Q_1$.

Using Schur Complement [12], the following inequalities are equivalent to each other

$$R_4 < 0 \Leftrightarrow (27)$$

If condition (27) holds, then $Z(t) = 0$ is uniformly asymptotically stable in (18). □

Remark 6: The stability criteria in Theorems 3–4 are delay independent and less conservative stability criteria for (18) may be found in other literature. Furthermore, delay-dependent stability criteria [19–20] can be also developed for (18) when a bound on the time delay τ is known. However, we focus on the tracking problem rather than the stabilisation problem of time-delay systems in this paper.

So readers may refer to other literature for less conservative stability criteria on (18).

4 Numerical simulations

Consider system (1) with parameter matrices as follows

Case 1:

$$A_0 = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\mathrm{T}}$$

$$A_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.15 & 0.1 \end{bmatrix}$$

Case 2:

$$A_0 = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^{\mathrm{T}}$$

$$A_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.15 & 0.1 \end{bmatrix}$$

where A_0, B, C are known a priori. A_1 is assumed unknown except $A_1 A_1^{\mathrm{T}} < 0.3 I_2$. $\tau = 5$ is the unknown time delay and $T = 2\pi$ is the period. The unknown periodic disturbance is $v(t) = \begin{bmatrix} \sin(t) & \sin^2(t) \end{bmatrix}^{\mathrm{T}}$. The control objective is to track given desired trajectories $y_{\mathrm{d},1}(t) = \sin(t)$ and $y_{\mathrm{d},2}(t)$, where $y_{\mathrm{d},2}(t)$ is a triangular waveform and the first period has the form as

$$y_{\mathrm{d},2}(t) = \begin{cases} \dfrac{2}{T}t, & 0 \le t < \dfrac{T}{2} \\ 2 - \dfrac{2}{T}t, & \dfrac{T}{2} \le t < T \end{cases}$$

4.1 Repetitive controller

In Case 1, by LMI control toolbox in MATLAB 6.5, the positive-definite solution of (9) subject to restriction $B^{\mathrm{T}} P = C$ is solved as $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\alpha = 0.62$. This implies that the stability of (25) is ensured and the controller (5) is designed as follows

$$u^{\mathrm{f}}(t) = u^{\mathrm{f}}(t - T) + K_0(t)\delta y(t)$$

$$K_1(t) = 3\frac{t}{T}, \quad K = 3 \tag{29}$$

This design choice makes $\int_{t-T}^{t} \| \delta y(\theta) \|^2 \mathrm{d}\theta$ tend to zero as $t \to +\infty$.

In Case 1, the tracking performance of system (1) driven by the controller (29) is shown in Fig. 2, where $\| \delta y_i \|_T \triangleq \sup_{t \in [0,T]} \| \delta y(iT + t) \|$.

The error is very small at the tenth period consistently with the result of Theorem 1. However, the conditions of
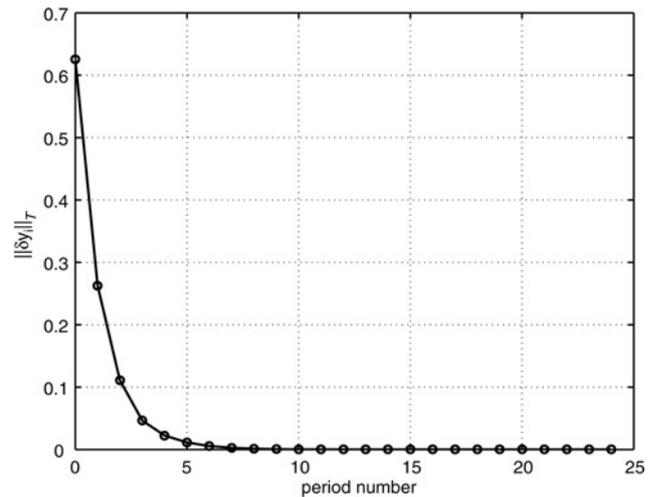
Figure 2 *Change of maximum absolute value of error at the ith period*

Theorem 1 do not hold for Case 2 or for the case when the desired trajectory $y_{\mathrm{d},2}(t)$ does not satisfy the appropriate differentiability condition (see Remark 1). These will be taken into consideration in Section 4.2.

4.2 Modified repetitive controller

In this simulation, the first-order filter $q(s)$ is chosen to be

$$q(s) = \frac{\omega_{\mathrm{c}}}{s + \omega_{\mathrm{c}}} \tag{30}$$

where $\omega_{\mathrm{c}} \in \mathbb{R}$ is the cutoff frequency of $q(s)$. Therefore the controller (16) is written as follows

$$\dot{x}_p(t) = -\omega_{\mathrm{c}} x_p(t) + \omega_{\mathrm{c}} x_p(t - T) + K \delta y(t)$$

$$u^{\mathrm{f}}(t) = \omega_{\mathrm{c}} x_p(t - T) + K \delta y(t) \tag{31}$$

Since the stability of (25) has been ensured in Section 4.1, we only need to consider the stability of (18) in this section.

In Case 2, when $\omega_{\mathrm{c}} = 60$, the positive-definite solution of (27) is solved by the LMI control toolbox as follows

$$Q_1 = \begin{bmatrix} 0.0413 & 0.0033 & 0.0012 \\ 0.0033 & 0.0043 & -0.0024 \\ 0.0012 & -0.0024 & 0.0864 \end{bmatrix}, \quad K = 3$$

$$Q_2 = \begin{bmatrix} 0.0464 & -0.0038 & 0.0175 \\ -0.0038 & 0.0064 & -0.0152 \\ 0.0175 & -0.0152 & 4.7121 \end{bmatrix}$$

$$\alpha = 0.0192$$

In Case 2, when we choose $\omega_{\mathrm{c}} = 60$, the tracking performance of the controller (31) with different desired trajectories $y_{\mathrm{d},1}(t)$, $y_{\mathrm{d},2}(t)$ is depicted in Fig. 3.
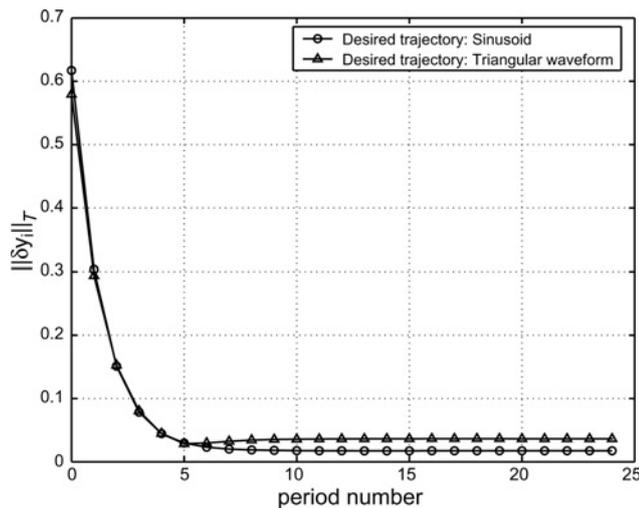
Figure 3 *Change of maximum absolute value of error at the ith period*

Since the amplitude of desired trajectories $y_{d,1}(t)$ and $y_{d,2}(t)$ is the same and C_2 in inequality (24) is independent of $y_{d,1}(t)$ and $y_{d,2}(t)$, the tracking performance is determined by ϵ in inequality (24). In this case since the triangular waveform $y_{d,2}(t)$ has more high-frequency components than sinusoid $y_{d,1}(t)$, ϵ is smaller when using sinusoid $y_{d,1}(t)$ as the desired trajectory. Therefore the tracking performance under desired trajectory $y_{d,1}(t)$ is better than that under desired trajectory $y_{d,2}(t)$ as observed in Fig. 3.

Fig. 4 depicts the tracking performance of the controller (31) with different cutoff frequencies $\omega_c = 10, 50, 90$ in Case 2, when using the desired trajectory $y_{d,1}(t)$.

Remark 7: As observed in Fig. 4, the best tracking performance is achieved when $\omega_c = 90$. It is noteworthy that there is little difference between the tracking
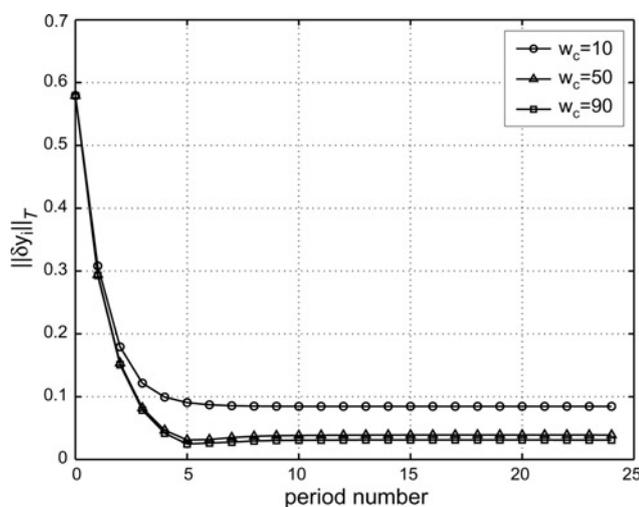
Figure 4 *Change of maximum absolute value of error at the ith period*

performance under $\omega_c = 50$ and $\omega_c = 90$. In this simulation, the stability margin decreases as ω_c in (30) increases, which leads to the decrease of ϵ and increase of C_2 in (24). Therefore $C_2\epsilon$ in (24) may change slightly when ω_c is large enough. In practice, it is difficult to determine the variation of the value $C_2\epsilon$ and the tracking performance with different ω_c. Nevertheless, we can adjust ω_c through observing the tracking performance in practice.

Remark 8: With respect to tracking performance, the controller presented in Theorem 1 outperforms that in Theorems 3–4. However, the controller presented in Theorems 3–4, which has fewer restrictions, can be applied to more general systems.

5 Conclusions

This paper aims to design output error controllers to track periodic reference signals for a class of uncertain linear state-delayed systems subject to periodic disturbances. The novel contributions of this paper are as follows: (i) controllers are designed and analysed for a class of uncertain linear state-delayed systems to track periodic reference signals; (ii) only the output error signal is available for feedback and the proposed controllers do not require the time derivatives of the output error; (iii) the theory on the modified RC system is extended to apply to the systems represented by irrational transfer functions; (iv) most of the criteria in this paper are given in terms of LMIs that make them quite feasible in the controller/filter design with the aid of a computer.

6 Acknowledgment

7 References

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