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A new model transformation method and its application to extending a class of stability criteria of neutral type systems

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ABSTRACT

This paper proposes a generalized equivalent model transformation method, which can include methods proposed by Fridman et al. and Bellen et al., for the stability analysis of a class of neutral type systems. By using the proposed model transformation method, a class of existing stability criteria derived by the Lyapunov functional approach can be extended to less conservative ones in terms of nonlinear matrix inequalities. Furthermore, procedures to solve these nonlinear matrix inequalities are also proposed. Illustrative examples are presented to demonstrate the effectiveness of the proposed model transformation method.

1. Introduction

Recently, many model transformation methods have been proposed for the stability analysis of neutral type systems [1–3]. However, some of the transformed systems are not equivalent to the original systems. In order to overcome the problem, Fridman et al. proposed an equivalent augmented model as a "descriptor form" representation of the system [2,3]. Another similar equivalent model transformation method was proposed by Bellen et al. [4].

Both models proposed by Fridman et al. and Bellen et al. are in fact special cases of the 2D model [5–7]. Based on this fact, we introduce a slack matrix into the equivalent augmented model proposed by Fridman et al. to form a generalized 2D model. Through choosing specific slack matrices, the proposed generalized model can recover equivalent models proposed by Fridman et al. and Bellen et al. The new model transformation method can be used to extend many existing stability criteria. However, in order to demonstrate the effectiveness more explicitly, we only focus on extending a class of existing stability criteria. By using this new model transformation method and the Lyapunov functional approach, the class of existing stability criteria can be extended to less conservative criteria in terms of nonlinear matrix inequalities. In view of this, procedures to solve these nonlinear matrix inequalities are also proposed. The effectiveness of the proposed model transformation method is demonstrated through illustrative examples. The main contributions of this paper are: (1) a new model transformation method is proposed; (2) based on this new transformation method, this paper extends a class of existing stability criteria rather than just designing new Lyapunov functionals, and it is proven that the extended criteria can reduce the conservatism of the original criteria; (3) procedures to solve a class of nonlinear matrix inequalities are proposed.

The notation used in this paper is as follows. $\mathbb{R}^n$ is the Euclidean space of dimension n. X^T is the transpose of matrix X. $|\cdot|$ denotes the absolute value of a scalar and $\|\cdot\|$ denotes the Euclidean norm or the matrix norm induced by the Euclidean norm. $\lambda_{\max}(X)$ denotes the maximal eigenvalue of matrix X. $X > 0$ $(X < 0)$ represents that matrix X is a positive definite (negative definite) matrix. I_n is an identity matrix of a specified dimension n. $0_{n\times m} \in \mathbb{R}^{n\times m}$ denotes a zero matrix.

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2. A new model transformation method

Given the following neutral type system:

$$0 = F(A_0, A_1, x, \dot{x}, t) \tag{1}$$

with the initial condition:

$$x(t) = \phi(t), \quad \forall t \in [-\tau, 0]$$

where

$$F(X_0, X_1, x, v, t) \triangleq v(t) - Dv(t-\tau) - X_0 x(t) - X_1 x(t-\tau) - f(x,t)$$
$$x(t), v(t) \in \mathbb{R}^{n\times 1}, X_0, X_1 \in \mathbb{R}^{n\times n}, f(x,t) \in \mathbb{R}^{n\times 1}, \tau \in \mathbb{R}.$$

In (1), $X_0 = A_0, X_1 = A_1, v(t) = \dot{x}(t)$. $x(t)$ is the state vector, τ is the time delay, $f(x,t)$ is a vector function which does not contain derivative terms of the state vector, $\phi(t)$ is a continuously differentiable smooth vector valued function representing the initial condition function. If $D = 0$, then system (1) degenerates into a retarded type system. Without loss of generality, the purpose of this paper is to propose a new model transformation method to derive stability criteria for neutral type system (1).

Define

$$y(t) \triangleq \dot{x}(t) - Sx(t) \tag{2}$$

where $S \in \mathbb{R}^{n\times n}$ is the slack matrix which needs to be designed to obtain less conservative stability criteria. This will be discussed later.

Since the vector function $f(x,t)$ does not contain derivative terms of the state vector, substituting $\dot{x}(t) = Sx(t) + y(t)$ into (1) yields:

$$0 = [Sx(t) + y(t)] - D[Sx(t-\tau) + y(t-\tau)] - A_0 x(t) - A_1 x(t-\tau) - f(x,t).$$

Consequently, system (1) is transformed into the following equivalent form:

$$\begin{cases} \dot{x}(t) = Sx(t) + y(t) \\ 0 = F(A_0 - S, A_1 + DS, x, y, t) \end{cases} \tag{3}$$

where $y(t)$ can be treated as the 'fast variable' as mentioned in [2].

Since the transformed system (3) is equivalent to the original system, we will focus on the stability analysis of the transformed system in the following sections. When choosing $S = 0$ or $S = A_0$, system (3) can recover the equivalent models proposed in [2] or [4], respectively. Furthermore, by designing the appropriate slack matrix S, the conservatism of the criteria derived by the model transformation methods proposed in [2,4] can be effectively reduced.

3. A method to extend a class of stability criteria

The proposed model transformation method can help to design new Lyapunov functionals and then obtain new stability criteria as proposed in [2]. However, in order to demonstrate the effectiveness more explicitly, we only focus on extending a class of existing stability criteria in this section. First, by applying the proposed model transformation method, a simple application on extending an existing stability criterion is given. Following the idea of this application, a generalized method in terms of a theorem is derived to extend a class of existing stability criteria. Finally, a stability criterion proposed in [8] is extended to a less conservative one by using the generalized method.

3.1. A simple application

For simplicity, consider neutral type system (1) with $f(x,t) \equiv 0$, i.e.

$$0 = F_1(A_0, A_1, x, \dot{x}, t) \tag{4}$$

where $F_1(X_0, X_1, x, v, t) \triangleq v(t) - Dv(t-\tau) - X_0 x(t) - X_1 x(t-\tau)$.

The following criterion proposed in [2] is used to determine the delay-independent stability of neutral type system (4).

Fact 1 ([2]). *If there exist* $0 < P_1 = P_1^T, P_2, P_3 \in \mathbb{R}^{n\times n}$ *and* $0 < Q_1 = Q_1^T, 0 < Q_2 = Q_2^T \in \mathbb{R}^{n\times n}$ *such that:*

$$\Omega_1(A_0, A_1, \mathcal{P}_1) < 0 \tag{5}$$

then the solution $x(t,\phi)$ of neutral type system (4) is asymptotically stable, where

$$\Omega_1(X_0,X_1,\mathcal{P}_1) \triangleq \begin{bmatrix} X_0^T P_2 + P_2^T X_0 + Q_2 & P_1 - P_2^T + X_0^T P_3 & P_2^T X_1 & P_2^T D \\ P_1 - P_2 + P_3^T X_0 & -P_3^T - P_3 + Q_1 & P_3^T X_1 & P_3^T D \\ X_1^T P_2 & X_1^T P_3 & -Q_2 & 0_{n\times n} \\ D^T P_2 & D^T P_3 & 0_{n\times n} & -Q_1 \end{bmatrix}$$

$$\mathcal{P}_1 = \begin{bmatrix} P_1 & P_2 & P_3 & Q_1 & Q_2 \end{bmatrix}.$$

Fact 1 is a special case of Corollary 2 proved in [2]. The outline of the proof of Fact 1 is described as follows. The Lyapunov functional is chosen to be $V_1(x,\dot{x},t)$, where

$$V_1(x,v,t) \triangleq x(t)^T P_1 x(t) + \int_{t-\tau}^{t} v(s)^T Q_1 v(s)\,ds + \int_{t-\tau}^{t} x(s)^T Q_2 x(s)\,ds. \tag{6}$$

The time derivative of $V_1(x,\dot{x},t)$ is

$$\dot{V}_1(x,\dot{x},t) = 2x(t)^T P_1 \dot{x}(t) + \dot{x}(t)^T Q_1 \dot{x}(t) - \dot{x}(t-\tau)^T Q_1 \dot{x}(t-\tau) + x(t)^T Q_2 x(t) - x(t-\tau)^T Q_2 x(t-\tau).$$

Introducing a zero term

$$\Xi_1(x,\dot{x},t)^T F_1(A_0,A_1,x,\dot{x},t) = 0 \tag{7}$$

into equation above yields

$$\begin{aligned}\dot{V}_1(x,\dot{x},t) &= 2x(t)^T P_1 \dot{x}(t) + \dot{x}(t)^T Q_1 \dot{x}(t) - \dot{x}(t-\tau)^T Q_1 \dot{x}(t-\tau) + x(t)^T Q_2 x(t) - x(t-\tau)^T Q_2 x(t-\tau) \\ &\quad + \Xi_1^T(x,\dot{x},t) F_1(A_0,A_1,x,\dot{x},t) \\ &= \Sigma_1(A_0,A_1,x,\dot{x},t)\end{aligned}$$

where

$$\begin{aligned}&\Xi_1(x,v,t) \triangleq P_2 x(t) + P_3 v(t) \\ &\Sigma_1(X_0,X_1,x,v,t) \triangleq \xi_1(x,v,t)^T \Omega_1(X_0,X_1,\mathcal{P}_1)\,\xi_1(x,v,t) \\ &\xi_1(x,v,t) \triangleq \begin{bmatrix} x(t)^T & v(t)^T & x(t-\tau)^T & v(t-\tau)^T \end{bmatrix}^T.\end{aligned}$$

This leads to Fact 1. Now, we will apply transformed system (3) to the proof of Fact 1 and obtain an extended stability criterion.

Design a new Lyapunov functional $V_1(x,y,t)$, where v in (6) is replaced by y. The time derivative of $V_1(x,y,t)$ is

$$\begin{aligned}\dot{V}_1(x,y,t) &= 2x(t)^T P_1 y(t) + y(t)^T Q_1 y(t) - y(t-\tau)^T Q_1 y(t-\tau) \\ &\quad + x(t)^T Q_2 x(t) - x(t-\tau)^T Q_2 x(t-\tau) + 2x(t)^T P_1 S x(t)\end{aligned}$$

where $\dot{x}(t) = Sx(t) + y(t)$ is used. Introducing a zero term

$$\Xi_1(x,y,t)^T F_1(A_0 - S, A_1 + DS, x, y, t) = 0 \tag{8}$$

into equation above yields

$$\begin{aligned}\dot{V}_1(x,y,t) &= 2x(t)^T P_1 y(t) + y(t)^T Q_1 y(t) - y(t-\tau)^T Q_1 y(t-\tau) + x(t)^T Q_2 x(t) - x(t-\tau)^T Q_2 x(t-\tau) \\ &\quad + 2x(t)^T P_1 S x(t) + \Xi_1(x,y,t)^T F(A_0 - S, A_1 + DS, x, y, t) \\ &= \Sigma_1(A_0 - S, A_1 + DS, x, y, t) + 2x(t)^T P_1 S x(t).\end{aligned} \tag{9}$$

This leads to Corollary 1 which is an extended result of Fact 1.

Corollary 1. *If there exist $0 < P_1 = P_1^T, P_2, P_3 \in \mathbb{R}^{n\times n}$, $0 < Q_1 = Q_1^T, 0 \le Q_2 = Q_2^T \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$ such that:*

$$\Omega_1(A_0 - S, A_1 + DS, \mathcal{P}_1) + \tilde{\Omega}_1(S,\mathcal{P}_1) < 0, \tag{10}$$

then the solution $x(t,\phi)$ of neutral type system (4) is asymptotically stable, where

$$\tilde{\Omega}_1(S,\mathcal{P}_1) = \begin{bmatrix} P_1 S + S^T P_1 & 0_{n\times 3n} \\ * & 0_{3n\times 3n} \end{bmatrix}.$$

Remark 1. If the original criterion (5) in Fact 1 has a feasible solution, denoted by $\mathcal{P}^*$, then there also exists a feasible solution, i.e. $S = 0$ and $\mathcal{P}_1 = \mathcal{P}_1^*$, to the extended criterion (10) in Corollary 1. However, this does not hold vice versa, i.e. there may not exist a feasible solution to the original criterion in Fact 1 when the extended criterion (10) has a feasible solution. Therefore, the extended criterion is less conservative than the original criterion.

3.2. A method to extend a class of stability criteria

In the preceding section, we use y to play the role of $\dot{x}$. Compared with $\dot{x}$, y has a freedom to choose the slack matrix S. An important step of obtaining the extended criterion is to substitute the zero term (8) for (7) in the proof of Corollary 1, where transformed system (3) is utilized.

Based on the idea of the application above, we will extend a class of existing stability criteria in this section. To begin with, we need some properties for the existing stability criteria which need to be extended.

For the stability of neutral type system (1), a normative proof process is described as follows:

Property 1. (i) *A nonnegative functional $V(x, \dot{x}, t)$ is designed as*

$$V(x, \dot{x}, t) = V_m(x, \dot{x}, t) + V_a(x, \dot{x}, t)$$

where

$$V_m(x, v, t) \triangleq \eta_0(t)\, x(t)^T P_0 x(t) + \int_{t-\tau}^{t} \eta_1(s)\, x(s)^T P_1 x(s)\, ds + \int_{t-\tau}^{t} \eta_2(s)\, v(s)^T P_2 v(s)\, ds$$

$$V_a(x, v, t) \triangleq \int_{-\tau}^{0}\int_{t+\theta}^{t} \eta_3(s)\, v^T(s) P_3 v(s)\, ds d\theta$$

and $0 < P_k = P_k^T \in \mathbb{R}^{n\times n}$ *and* $\eta_k(t) \in \mathbb{R}$ *are nonnegative functions,* $k = 0, 1, 2, 3$.

(ii) *The time derivative of $V(x, \dot{x}, t)$ in* (i) *is calculated as follows:*

$$\dot{V}(x, \dot{x}, t) = \partial_t V_m(x, \dot{x}, t) + \partial_x V_m(x, \dot{x}, t)^T \dot{x}(t) + \partial_t V_a(x, \dot{x}, t)$$

where $\partial_t V \triangleq \frac{\partial V}{\partial t} \in \mathbb{R}^{n\times 1}$ *and* $\partial_x V \triangleq \frac{\partial V}{\partial x} \in \mathbb{R}^{n\times 1}$. *Along a given trajectory of system* (1), $F(A_0, A_1, x, \dot{x}, t) = 0$ *holds. By introducing zero terms*

$$\Xi(x, \dot{x}, t)^T F(A_0, A_1, x, \dot{x}, t) = 0$$
$$\Upsilon(x, \dot{x}, t)^T \kappa_z(x, \dot{x}, t) = 0$$

and a nonnegative term

$$\kappa_{nn}(x, t) \geq 0,$$

$\dot{V}(x, \dot{x}, t)$ *can be further written as*

$$\dot{V}(x, \dot{x}, t) \leq \Pi_m(A_0, A_1, x, \dot{x}, t) + \partial_t V_a(x, \dot{x}, t) \quad (11)$$
$$\Pi_m(X_0, X_1, x, v, t) \triangleq \partial_t V_m(x, v, t) + \partial_x V_m(x, v, t)^T v(t) + \Xi(x, v, t)^T F(X_0, X_1, x, v, t) + \Upsilon(x, v, t)^T \kappa_z(x, v, t) + \kappa_{nn}(x, t)$$

where $\Xi(x, v, t) \in \mathbb{R}^{n\times 1}$, $\kappa_z(x, v, t) \in \mathbb{R}^{n\times 1}$, $\kappa_z(x, \dot{x}, t) \equiv 0$ *for* $\forall x, \dot{x} \in \mathbb{R}^{n\times 1}$ *and* $\kappa_{nn}(x, t) \in \mathbb{R}$, $\kappa_{nn}(x, t) \geq 0$ *for* $\forall x \in \mathbb{R}^{n\times 1}$.

(iii) *For* $\forall x, \dot{x} \in \mathbb{R}^{n\times 1}$, *inequality* (12) *holds*

$$\partial_t V_a(x, \dot{x}, t) \leq \Pi_a(x, \dot{x}, t). \quad (12)$$

By using (12), (11) *becomes*

$$\dot{V}(x, \dot{x}, t) \leq \Sigma(A_0, A_1, x, \dot{x}, t) \quad (13)$$

where $\Sigma(X_0, X_1, x, v, t) \triangleq \Pi_m(X_0, X_1, x, v, t) + \Pi_a(x, v, t)$.

Remark 2. For simplicity, we introduce the form of nonnegative functional $V(x, \dot{x}, t)$ as in Property 1(i). By applying Property 1, we aim to show how to use the new model transformation in a stability proof. In fact, the form can be changed according to actual situation and then the following results will be changed correspondingly. The proof of Fact 1 is a normative proof process described in Property 1. Note that, in the proof of Fact 1, the terms as $V_a(x, \dot{x}, t)$, $\Upsilon(x, \dot{x}, t)^T \kappa_z(x, \dot{x}, t)$ and $\kappa_{nn}(x, t)$ are not introduced.

Remark 3. Some proof processes of existing stability criteria seem to be slightly different from the normative process described in Property 1. But if these proof processes can be normalized as the process described in Property 1, then Property 1 still holds for them. Take system $\dot{x}(t) = h(x, t)$ for example, the time derivative of $V(x, \dot{x}, t)$ along a given trajectory of system $\dot{x}(t) = h(x, t)$ is calculated as follows:

$$\dot{V}(x, \dot{x}, t) = \partial_t V(x, \dot{x}, t) + \partial_x V(x, \dot{x}, t)^T h(x, t). \quad (14)$$

The method of introducing zero terms or nonnegative terms is not formally used as (11) in Property 1. In fact, along a given trajectory of system $\dot{x}(t) = h(x, t)$, $\dot{V}(x, \dot{x}, t)$ can also be rewritten as

$$\dot{V}(x, \dot{x}, t) = \partial_t V(x, \dot{x}, t) + \partial_x V(x, \dot{x}, t)^T \dot{x}(t) + \Xi(x, \dot{x}, t)^T [\dot{x} - h(x, t)]. \tag{15}$$

If $\Xi(x, \dot{x}, t) = -\partial_x V(x, \dot{x}, t)$ is chosen, then the above equation becomes (14). This implies that the derivation process (14) is only a special case of (15).

Remark 4. Leibniz–Newton formula [9,10] is often used to construct the zero term $\kappa_z(x, \dot{x}, t)$. For example, $\kappa_z(x, \dot{x}, t) = x(t) - x(t-\tau) - \int_{t-\tau}^{t} \dot{x}(s)\,ds \equiv 0$.

Theorem 1. *Suppose Property 1 holds for (1). The time derivative of a given Lyapunov functional*

$$V(x, \dot{x}, y, t) = V_m(x, y, t) + V_a(x, \dot{x}, t)$$

along a given trajectory of system (3) is calculated as follows:

$$\dot{V}(x, \dot{x}, y, t) \le \Sigma(A_0 - S, A_1 + DS, x, y, t) + \Phi(S, x, y, t) \tag{16}$$

where

$$\begin{aligned}\Phi(S, x, y, t) = {} & \partial_x V_m(x, y, t)^T Sx(t) + [\Pi_a(x, Sx + y, t) - \Pi_a(x, y, t)] \\ & + \Upsilon(x, y, t)^T [\kappa_z(x, Sx + y, t) - \kappa_z(x, y, t)].\end{aligned} \tag{17}$$

Proof. The time derivative of $V(x, \dot{x}, y, t)$ is calculated as follows:

$$\dot{V}(x, \dot{x}, y, t) = \partial_t V_m(x, y, t) + \partial_x V_m(x, y, t)^T \dot{x}(t) + \partial_t V_a(x, \dot{x}, t).$$

Substituting $\dot{x}(t) = Sx(t) + y(t)$ (see (2)) into the above equation yields

$$\dot{V}(x, \dot{x}, y, t) = \partial_t V_m(x, y, t) + \partial_x V_m(x, y, t)^T y(t) + \partial_t V_a(x, \dot{x}, t) + \partial_x V_m(x, y, t)^T Sx(t). \tag{18}$$

According to (11), by introducing the zero terms

$$\Xi(x, y, t)^T F(A_0 - S, A_1 + DS, x, y, t) = 0$$
$$\Upsilon(x, y, t)^T \kappa_z(x, \dot{x}, t) = 0$$

and the nonnegative term

$$\kappa_{nn}(x, t) \ge 0,$$

Eq. (18) can be further rewritten as

$$\dot{V}(x, \dot{x}, y, t) \le \Pi_m(A_0 - S, A_1 + DS, x, y, t) + \partial_t V_a(x, \dot{x}, t) + \tilde{\Phi}(S, x, y, t) \tag{19}$$

where

$$\tilde{\Phi}(S, x, y, t) = \partial_x V_m(x, y, t)^T Sx(t) + \Upsilon(x, y, t)^T [\kappa_z(x, Sx + y, t) - \kappa_z(x, y, t)].$$

Consequently, based on (12) and (13) in Property 1, the inequality (19) becomes (16). It completes this proof. □

Remark 5. If the proof of an existing criterion is a normative proof process described in Property 1, then we only need to obtain $\Phi(S, x, y, t)$. Note that Π_m, not appearing in $\Phi(S, x, y, t)$, is only a intermediate variable, hence we do not need its concrete form. Recalling the outline of proof of Fact 1, the terms as $V_a(x, \dot{x}, t)$, $\Upsilon(x, \dot{x}, t)^T \kappa_z(x, \dot{x}, t)$ and $\kappa_{nn}(x, t)$ are not introduced. This implies

$$\Phi(S, x, y, t) = \partial_x V_m(x, y, t)^T Sx(t) = 2x(t)^T P_1 Sx(t).$$

Thus, (9) is consistent with (16).

Remark 6. Recalling the right-hand sides of (13) and (16), since

$$\min_{S \in \mathbb{R}^{n\times n}} \Sigma(A_0 - S, A_1 + DS, x, y, t) + \Phi(S, x, y, t) \le \Sigma(A_0, A_1, x, \dot{x}, t)$$
$$\Phi(0, x, y, t) \equiv 0$$

hence the conservatism of the extended criteria can be effectively reduced by choosing an appropriate slack matrix S. If choose $S = 0$, then $y(t) = x(t)$, $\Phi(S, x, y, t) = 0$ and, therefore, inequality (16) can recover inequality (13) in Property 1.

3.3. A case study

In order to further demonstrate the effectiveness of the proposed model transformation method and Theorem 1, let us consider neutral type system (1) with the nonlinear uncertainty $f(x,t) = g_1(x,t) + g_2(x,t)$ [8]:

$$0 = F_2(A_0, A_1, x, \dot{x}, t) \tag{20}$$

where $F_2(X_0, X_1, x, v, t) \triangleq v(t) - Dv(t-\tau) - X_0 x(t) - X_1 x(t-\tau) - g_1(x,t) - g_2(x,t)$. The functions $g_1(x,t) \in \mathbb{R}^{n\times 1}$ and $g_2(x,t) \in \mathbb{R}^{n\times 1}$ are nonlinear uncertainties which satisfy

$$g_1^T g_1 \le a^2 x(t)^T x(t), \qquad g_2^T g_2 \le b^2 x(t-\tau)^T x(t-\tau) \tag{21}$$

where $g_1 \triangleq g_1(x,t)$ and $g_2 \triangleq g_2(x,t)$.

Fact 2 ([8]). *Considering the uncertain nonlinear neutral system (20), for given scalars $\alpha > 0$ and $\tau > 0$, if there exist positive definite symmetric matrices $P, Q_1, Q_2, Q_3 \in \mathbb{R}^{n\times n}$ and positive scalars $\varepsilon_1, \varepsilon_2$ such that the following LMI holds:*

$$\Omega_2(A_0, A_1, \mathcal{P}_2) < 0 \tag{22}$$

then system (20) is robustly exponentially stable with decay rate α, where

$$\Omega_2(X_0, X_1, \mathcal{P}_2) \triangleq \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} \\ * & \Omega_{22} & \Omega_{23} & X_1^T(Q_2+\tau^2 Q_3) & X_1^T(Q_2+\tau^2 Q_3) \\ * & * & \Omega_{33} & D^T(Q_2+\tau^2 Q_3) & D^T(Q_2+\tau^2 Q_3) \\ * & * & * & -\varepsilon_1 I_n + Q_2 + \tau^2 Q_3 & Q_2+\tau^2 Q_3 \\ * & * & * & * & -\varepsilon_2 I_n + Q_2 + \tau^2 Q_3 \end{bmatrix}$$

$$\mathcal{P}_2 = \begin{bmatrix} P & Q_1 & Q_2 & Q_3 & \varepsilon_1 I_n & \varepsilon_2 I_n \end{bmatrix}$$

$$\Omega_{11} = PX_0 + X_0^T P + 2\alpha P + Q_1 + X_0^T(Q_2+\tau^2 Q_3)X_0 - e^{-2\alpha\tau}Q_3 + \varepsilon_1 a^2 I_n$$

$$\Omega_{12} = PX_1 + X_0^T(Q_2+\tau^2 Q_3)X_1 + e^{-2\alpha\tau}Q_3$$

$$\Omega_{13} = PD + X_0^T(Q_2+\tau^2 Q_3)D$$

$$\Omega_{14} = \Omega_{15} = P + X_0^T(Q_2+\tau^2 Q_3)$$

$$\Omega_{22} = -e^{-2\alpha\tau}Q_1 + X_1^T(Q_2+\tau^2 Q_3)X_1 - e^{-2\alpha\tau}Q_3 + \varepsilon_2 b^2 I_n$$

$$\Omega_{23} = X_1^T(Q_2+\tau^2 Q_3)D, \qquad \Omega_{33} = -e^{-2\alpha\tau}Q_2 + D^T(Q_2+\tau^2 Q_3)D.$$

The outline of the proof of Fact 2 is described as follows.

The Lyapunov functional $V_2(x,\dot{x},t)$ is chosen to be

$$V_2(x,\dot{x},t) = V_{2,m}(x,\dot{x},t) + V_{2,a}(x,\dot{x},t)$$

where

$$V_{2,m}(x,v,t) \triangleq e^{2\alpha t}x^T(t)Px(t) + \int_{t-\tau}^{t} e^{2\alpha s}x^T(s)Q_1 x(s)\,ds + \int_{t-\tau}^{t} e^{2\alpha s}v^T(s)Q_2 v(s)\,ds \tag{23}$$

$$V_{2,a}(x,v,t) \triangleq \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} e^{2\alpha s}v^T(s)Q_3 v(s)\,ds d\theta. \tag{24}$$

Therefore, Property 1(i) is satisfied. The term $\Upsilon(x,y,t)^T \kappa_z(x,\dot{x},t)$ is not introduced and Property 1(ii) is satisfied with

$$\Xi_2(x,\dot{x},t) = -2e^{2\alpha t}Px(t)$$

$$\kappa_{nn}(x,t) = \varepsilon_1 a^2 x(t)^T x(t) - \varepsilon_1 g_1^T g_1 + \varepsilon_2 b^2 x(t-\tau)^T x(t-\tau) - \varepsilon_2 g_2^T g_2$$

where (21) is utilized. The derivative of V_2 is bounded as

$$\dot{V}_2(x,\dot{x},t) \le \Pi_{2,m}(A_0, A_1, x, \dot{x}, t) + \partial_t V_{2,a}(x,\dot{x},t) \tag{25}$$

where the concrete form of $\Pi_{2,m}$ is not given here for simplicity. By using

$$-\tau\int_{t-\tau}^{t} \dot{x}(s)^T Q_3 \dot{x}(s)\,ds \le w(t)^T \begin{bmatrix} -Q_3 & Q_3 \\ Q_3 & -Q_3 \end{bmatrix} w(t),$$

the following inequality holds [8]

$$\begin{aligned}\partial_t V_{2,a}(x,\dot{x},t) &= e^{2\alpha t}\left[\tau^2 \dot{x}(t)^T Q_3 \dot{x}(t) - \tau \int_{t-\tau}^{t} e^{2\alpha(s-t)} \dot{x}^T(s) Q_3 \dot{x}(s)\,ds\right] \\ &\le \Pi_{2,a}(x,\dot{x},t)\end{aligned} \tag{26}$$

where

$$\begin{aligned}&\Pi_{2,a}(x,v,t) \triangleq e^{2\alpha t}\left\{\tau^2 v(t)^T Q_3 v(t) + e^{-2\alpha\tau} w(t)^T \begin{bmatrix} -Q_3 & Q_3 \\ Q_3 & -Q_3 \end{bmatrix} w(t)\right\} \\ &w(t) = \begin{bmatrix} x(t)^T & x(t-\tau)^T \end{bmatrix}^T.\end{aligned} \tag{27}$$

By using (26), inequality (25) becomes

$$\begin{aligned}\dot{V}_2(x,\dot{x},t) &\le \Pi_{2,m}(A_0,A_1,x,\dot{x},t) + \Pi_{2,a}(x,\dot{x},t) \\ &= \Sigma_2(A_0,A_1,x,\dot{x},t)\end{aligned}$$

where

$$\begin{aligned}&\Sigma_2(X_0,X_1,x,v,t) \triangleq e^{2\alpha t}\xi_2(x,v,t)^T \Omega_2(X_0,X_1,\mathcal{P}_2)\xi_2(x,v,t) \\ &\xi_2(x,v,t) \triangleq \begin{bmatrix} x(t)^T & x(t-\tau)^T & v(t-\tau)^T & g_1^T & g_2^T \end{bmatrix}^T.\end{aligned}$$

Therefore, Property 1(iii) is satisfied. The proof of Fact 2 is a normative proof process as described in Property 1.

The following Corollary 2 is an extended result of Fact 2 by Theorem 1.

Corollary 2. *Considering the uncertain nonlinear neutral system (20), for given scalars $\alpha > 0$ and $\tau > 0$, if there exist positive definite symmetric matrices $P, Q_1, Q_2, Q_3 \in \mathbb{R}^{n\times n}$, positive scalars $\varepsilon_1, \varepsilon_2$ and matrix $S \in \mathbb{R}^{n\times n}$ such that*

$$\Omega_2(A_0 - S, A_1 + DS, \mathcal{P}_2) + \tilde{\Omega}_2(S,\mathcal{P}_2) < 0 \tag{28}$$

then system (20) is robustly exponentially stable with decay rate α, where

$$\begin{aligned}&\tilde{\Omega}_2(S,\mathcal{P}_2) = \begin{bmatrix} \tilde{\Omega}_{11} & \tau^2 S^T Q_3 (A_1 + DS) & \tau^2 S^T Q_3 D & \tau^2 S^T Q_3 & \tau^2 S^T Q_3 \\ * & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ * & * & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ * & * & * & 0_{n\times n} & 0_{n\times n} \\ * & * & * & * & 0_{n\times n} \end{bmatrix} \\ &\tilde{\Omega}_{11} = PS + S^T P + \tau^2 S^T Q_3 S + \tau^2 (A_0 - S)^T Q_3 S + \tau^2 S^T Q_3 (A_0 - S).\end{aligned}$$

Proof. The candidate Lyapunov functional is chosen to be

$$V_2(x,\dot{x},y,t) = V_{2,m}(x,y,t) + V_{2,a}(x,\dot{x},t)$$

where $V_{2,m}(x,y,t)$ and $V_{2,a}(x,\dot{x},t)$ are defined in (23) and (24) respectively. By Theorem 1, $\dot{V}_2(x,\dot{x},y,t)$ satisfies

$$\dot{V}_2(x,\dot{x},y,t) \le \Sigma_2(A_0 - S, A_1 + DS, x, y, t) + \Phi_2(S,x,y,t). \tag{29}$$

The left problem is to obtain $\Phi_2(S,x,y,t)$. According to (27), we obtain

$$\Pi_{2,a}(x, Sx + y, t) - \Pi_{2,a}(x,y,t) = e^{2\alpha t}\tau^2\left[x(t)^T S^T Q_3 S x(t) + 2x(t)^T S^T Q_3 y(t)\right].$$

After substituting $y(t) = Dy(t-\tau) + (A_0 - S)x(t) + (A_1 + DS)x(t-\tau) + g_1 + g_2$ into the above equation, we can obtain

$$\begin{aligned}\Phi_2(S,x,y,t) &= 2e^{2\alpha t}x(t)^T PSx(t) + \left[\Pi_{2,a}(x, Sx + y, t) - \Pi_{2,a}(x,y,t)\right] \\ &= e^{2\alpha t}\xi_2(x,y,t)^T \tilde{\Omega}_2(S,\mathcal{P}_2)\xi_2(x,y,t).\end{aligned}$$

Consequently, (29) becomes

$$\dot{V}_2(x,\dot{x},y,t) \le e^{2\alpha t}\xi_2(x,y,t)^T\left[\Omega_2(A_0 - S, A_1 + DS, \mathcal{P}_2) + \tilde{\Omega}_2(S,\mathcal{P}_2)\right]\xi_2(x,y,t).$$

The subsequently proof is the same as in [8], so it is omitted here. □

4. Procedures to solve matrix inequalities

The extended stability criteria (10) and (28) are given in terms of existence of solutions to matrix inequalities such as

$$\Omega\left(A_0 - S, A_1 + DS, \mathcal{P}\right) + \tilde{\Omega}\left(S, \mathcal{P}\right) < 0$$

where $\tilde{\Omega}\left(0, \mathcal{P}\right) = 0$. When choose $S = 0$, the extended criteria (10) and (28) can recover the original criteria (5) in [2] and (22) in [8], respectively. Since A_0, A_1 and D are constant matrices, we redefine

$$\mathcal{M}\left(S, \mathcal{P}\right) \triangleq \Omega\left(A_0 - S, A_1 + DS, \mathcal{P}\right) + \tilde{\Omega}\left(S, \mathcal{P}\right).$$

In order to obtain less conservative, the slack matrix S can be determined by solving the following optimization problem

$$\begin{cases} \min\limits_{\mathcal{P}, S} \mu \\ \text{s.t. } \mathcal{M}\left(S, \mathcal{P}\right) < \mu I, \quad \mathcal{P} \in \mathcal{D}_{\mathcal{P}}, S \in \mathbb{R}^{n \times n} \end{cases} \tag{30}$$

where $\mathcal{D}_{\mathcal{P}}$ denotes the feasible domain of parameter $\mathcal{P}$.

Assume μ_{opt1} is the minimal value of (30) with restriction $S = 0$ and μ_{opt2} is the minimal value of (30). Obviously, $\mu_{opt2} \leq \mu_{opt1}$.$\mu_{opt1} < 0$ and $\mu_{opt2} < 0$ imply that original criteria and extended stability criteria are satisfied, respectively. If the original criteria have feasible solutions, i.e. there exists a $\mathcal{P} \in \mathcal{D}_{\mathcal{P}}$ such that $\mu_{opt1} < 0$, then there also exist feasible solutions to the extended criteria, i.e. $\mu_{opt2} < 0$. However, this does not hold vice versa, i.e. there may not exist feasible solutions to the original criteria when the extended criteria have feasible solutions. Therefore, the extended criteria are less conservative than the original criteria. The LMI Control Toolbox in MATLAB 6.5 cannot be used to solve the optimization problem (30) because $\mathcal{M}\left(S, \mathcal{P}\right)$ is a nonlinear matrix function with respect to S and $\mathcal{P}$. Therefore, calculation procedures need to be designed in order to obtain the optimal solution to the problem (30). For example:

$$\mathcal{M}_1\left(S, \mathcal{P}_1\right) = \Omega_1\left(A_0 - S, A_1 + DS, \mathcal{P}_1\right) + \tilde{\Omega}_1\left(S, \mathcal{P}_1\right) \tag{31}$$

$$\mathcal{M}_2\left(S, \mathcal{P}_2\right) = \Omega_2\left(A_0 - S, A_1 + DS, \mathcal{P}_2\right) + \tilde{\Omega}_2\left(S, \mathcal{P}_2\right) \tag{32}$$

where Ω_1, $\tilde{\Omega}_1$ and Ω_2, $\tilde{\Omega}_2$ are defined in (10) and (28), respectively. If matrix S is fixed, then $\mathcal{M}\left(S, \mathcal{P}\right)$ is a linear matrix function with respect to $\mathcal{P}$. On the other hand, when $\mathcal{P}$ is fixed, $\mathcal{M}\left(S, \mathcal{P}\right)$ may be a linear matrix function with respect to S as (31) or a nonlinear matrix function with respect to S as (32). The next step is to design calculation procedures to obtain suboptimal value μ^*, i.e. $\mu_{opt2} \leq \mu^* \leq \mu_{opt1}$, and corresponding suboptimal solution $\mathcal{P}^*$ and S^* with the aid of the LMI Control Toolbox.

4.1. Procedure 1

If $\mathcal{M}\left(S, \mathcal{P}\right)$ is a linear matrix function with respect to S as (31), similar to coordinate descent methods [11, pp. 53–55], the procedure to solve the optimization problem as (31) is designed as follows:

Procedure 1	
Step 1:	Set $S_0 = 0$, $k = -1$, $\varepsilon\ (\varepsilon > 0)$, $\mathcal{D}_{\mathcal{P}_1}$.
Step 2:	$k = k + 1$. Solve the following optimization problem $\begin{cases} \min\limits_{\mathcal{P}} \mu \\ s.t.\ \mathcal{M}\left(S_k, \mathcal{P}\right) \leq \mu I, \mathcal{P} \in \mathcal{D}_{\mathcal{P}_1} \end{cases} \quad (*)$ and obtain the optimal solution $\mathcal{P}_k^*$ and the optimal value μ_k. If $\mu_k < 0$, then output $k^* = k$, $\mu^* = \mu_k$, $\mathcal{P}^* = \mathcal{P}_k^*$, $S^* = S_k$ and step out. Otherwise go to *Step 3.*
Step 3:	Solve the following optimization problem $\begin{cases} \min\limits_{S} \varsigma \\ s.t.\ \mathcal{M}\left(S, \mathcal{P}_k^*\right) \leq \varsigma I \end{cases} \quad (**)$ and obtain the optimal solution S_k^* and the optimal value ς_k. If $\varsigma_k < 0$, then output $k^* = k$, $\mu^* = \mu_k$, $\mathcal{P}^* = \mathcal{P}_k^*$, $S^* = S_k^*$, and step out. Otherwise go to *Step 4.*
Step 4:	If $\left|\left(\mu_k - \varsigma_k\right)/\left(\mu_k + \varsigma_k\right)\right| < \varepsilon$, then $k^* = k$ and step out. Otherwise, set $S_{k+1} = S_k^*$ and go to *Step 2.*

In *Step 2* of Procedure 1, S is fixed to find a new solution $\mathcal{P}$ that minimizes the objective function of $(*)$. In *Step 3*, $\mathcal{P}$ is fixed to repeat the process for S, and so on. Since $\mathcal{M}\left(S, \mathcal{P}\right)$ is a linear matrix function with respect to S when the other variable is fixed (vice versa), optimization problems $(*)$ and $(**)$ in Procedure 1 can be solved by using function "gevp" in the LMI Control Toolbox. Obviously, $\mu_{opt2} \leq \mu^* = \mu_{k^*} \leq \varsigma_{k^*-1} \leq \mu_{k^*-1} \cdots \leq \varsigma_0 \leq \mu_0 \leq \mu_{opt1}$ holds, hence μ^* is the suboptimal solution.

Example 1. Consider a simple system (4) with

$$D=\begin{bmatrix}0.4 & 0\\ 0 & -0.4\end{bmatrix},\qquad A_0=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix},\qquad A_1=\begin{bmatrix}0.1 & 0.1\\ 0.1 & 0.1\end{bmatrix}.$$

For the above example, when choosing $S=0$ in criterion (10), i.e. using criterion (5) in [2], there does not exist a feasible solution. Note that the extended criterion (10) still does not have a feasible solution when choosing $S=A_0$. Therefore we follow Procedure 1 to find an appropriate S such that criterion (10) has a feasible solution. Let $\varepsilon=0.001$,

$$\mathcal{D}_{\mathcal{P}_1}=\left\{\mathcal{P}_1\left|\begin{matrix}0.1I_2\le P_1\le 100I_2 & 0.1I_2\le Q_1\le 100I_2\\ 0.1I_2\le Q_2\le 100I_2 & P_1,P_2,P_3,Q_1,Q_2\in\mathbb{R}^{2\times 2}\end{matrix}\right.\right\}$$

and following Procedure 1 for criterion (10), we obtain $k^*=1$, $\mu^*=-1.3802$ and

$$S^*=\begin{bmatrix}0.0727 & 0.1653\\ -0.1218 & -0.0912\end{bmatrix}\quad P_2^*=\begin{bmatrix}5.08 & -1.04\\ 4.31 & 4.48\end{bmatrix}\quad P_1^*=\begin{bmatrix}6.23 & 2.38\\ 2.38 & 6.79\end{bmatrix}$$
$$P_3^*=\begin{bmatrix}6.74 & -1.49\\ 3.16 & 5.70\end{bmatrix}\quad Q_1^*=\begin{bmatrix}3.78 & -0.23\\ -0.23 & 2.69\end{bmatrix}\quad Q_2^*=\begin{bmatrix}0.99 & 0.44\\ 0.44 & 1.45\end{bmatrix}.$$

Therefore, system (4) in Example 1 is asymptotically stable. Procedure 1 cannot be used for criterion (32), because, in this case, $\mathcal{M}_2(S,\mathcal{P})$ is a nonlinear matrix function with respect to S when $\mathcal{P}$ is fixed. In view of this, Procedure 2 is proposed.

4.2. Procedure 2

$\mathcal{M}(S+\Delta S,\mathcal{P})$ can be written as

$$\mathcal{M}(S+\Delta S,\mathcal{P})=\mathcal{M}(S,\mathcal{P})+\mathcal{L}(\Delta S,S,\mathcal{P})+o\left(\|\Delta S\|^2\right)$$

where $\mathcal{M}(S,\mathcal{P})$ does not include ΔS, $\mathcal{L}(\Delta S,S,\mathcal{P})$ is a linear matrix function with respect to ΔS when S and $\mathcal{P}$ are fixed and $\lim_{\|\Delta S\|\to 0}\left\|o\left(\|\Delta S\|^2\right)\right\|/\|\Delta S\|=0$. For example, if

$$\mathcal{M}(S+\Delta S,\mathcal{P})=\begin{bmatrix}P(S+\Delta S)+(S+\Delta S)^T P & S+\Delta S\\ (S+\Delta S)^T & -(S+\Delta S)^T P(S+\Delta S)\end{bmatrix}$$

where $P,S,\Delta S\in\mathbb{R}^{n\times n}$, then

$$\mathcal{M}(S,\mathcal{P})=\begin{bmatrix}PS+S^TP & S\\ S^T & -S^TPS\end{bmatrix}$$
$$\mathcal{L}(\Delta S,S,\mathcal{P})=\begin{bmatrix}P\Delta S+\Delta S^TP & \Delta S\\ \Delta S^T & -\Delta S^TPS-S^TP\Delta S\end{bmatrix}$$
$$o\left(\|\Delta S\|^2\right)=\begin{bmatrix}0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & -\Delta S^TP\Delta S\end{bmatrix}.$$

For (32), we have

$$\mathcal{M}_2(S+\Delta S,\mathcal{P}_2)=\mathcal{M}_2(S,\mathcal{P}_2)+\mathcal{L}_2(\Delta S,S,\mathcal{P}_2)+o\left(\|\Delta S\|^2\right)$$

where

$$\mathcal{L}_2(\Delta S,S,\mathcal{P}_2)=\begin{bmatrix}L_{11} & L_{12} & -\Delta S^TQ_2D & -\Delta S^TQ_2 & -\Delta S^TQ_2\\ * & L_{22} & L_{23} & L_{24} & L_{25}\\ * & * & 0_{n\times n} & 0_{n\times n} & 0_{n\times n}\\ * & * & * & 0_{n\times n} & 0_{n\times n}\\ * & * & * & * & 0_{n\times n}\end{bmatrix}$$
$$L_{11}=-\Delta S^TQ_2(A_0-S)-(A_0-S)^TQ_2\Delta S$$
$$L_{12}=-\Delta S^TQ_2(A_1+DS)+\left[PD+(A_0-S)^TQ_2D+\tau^2A_0^TQ_3D\right]\Delta S$$
$$L_{22}=\Delta S^TD^T\left(Q_2+\tau^2Q_3\right)(A_1+DS)+(A_1+DS)^T\left(Q_2+\tau^2Q_3\right)D\Delta S$$
$$L_{23}=\Delta S^TD^T\left(Q_2+\tau^2Q_3\right)D$$
$$L_{24}=L_{25}=\Delta S^TD^T\left(Q_2+\tau^2Q_3\right).$$

We try to obtain ΔS with the aid of LMI Control Toolbox at each iteration in order to update S. Define a region $\|\Delta S\|^2\le\epsilon$ around the current iterate of S within which the linear matrix function $\mathcal{M}(S,\mathcal{P})+\mathcal{L}(\Delta S,S,\mathcal{P})$ can be trusted to be an adequate representation of $\mathcal{M}(S + \Delta S, \mathcal{P})$. Similar to the idea of trust-region methods [11, pp. 65–68], the procedure to solve the optimization problem (32) is designed as follows:

Procedure 2	
Step 1:	Set $S_0 = 0$, $k = -1$, $\varepsilon(\varepsilon > 0)$, ϵ_0, $\mathcal{D}_{\mathcal{P}_2}$, η_v, η_s, γ_d, γ_i
Step 2:	$k = k + 1$. Solve the following optimization problem $\begin{cases} \min\limits_{\mathcal{P}} \mu \\ s.t.\ \mathcal{M}(S_k, \mathcal{P}) \le \mu I,\ \mathcal{P} \in \mathcal{D}_{\mathcal{P}_2} \end{cases} \quad (*)$ and obtain the optimal solution $\mathcal{P}_k^*$ and the optimal value μ_k. If $\mu_k < 0$, then output $k^* = k$, $\mu^* = \mu_k$, $\mathcal{P}^* = \mathcal{P}_k^*$, $S^* = S_k$, and step out. Otherwise go to *Step 3.*
Step 3:	Solve the following optimization problem $\begin{cases} \min\limits_{\Delta S} \varsigma \\ s.t.\ \mathcal{M}\left(S_k, \mathcal{P}_k^*\right) + \mathcal{L}\left(\Delta S, S_k, \mathcal{P}_k^*\right) \le \varsigma I,\ \begin{bmatrix} -\epsilon_k I_n & \Delta S \\ \Delta S^T & -I_n \end{bmatrix} \le 0 \end{cases} \quad (**)$ and obtain the optimal solution ΔS_k^* and the optimal value ς_k. Go to *Step 4*.
Step 4:	(i) If $\lvert(\mu_k - \varsigma_k)/(\mu_k + \varsigma_k)\rvert < \varepsilon$, then $k^* = k$ and step out. Set $\rho = (\mu_k - \bar{\mu}_k)/(\mu_k - \varsigma_k)$, where $\bar{\mu}_k = \mathcal{M}\left(S_k + \Delta S_k^*, \mathcal{P}_k^*\right)$. (ii) Otherwise if $\rho \ge \eta_v$ [very successful, $0 < \eta_v < 1$], then set $S_{k+1} = S_k + \Delta S_k^*$ and $\epsilon_{k+1} = \gamma_i \epsilon_k$, where $\gamma_i \ge 1$. Go to *Step 2.* (iii) Otherwise if $\rho \ge \eta_s$ [successful, $0 < \eta_s \le \eta_v < 1$], then set $S_{k+1} = S_k + \Delta S_k^*$ and $\epsilon_{k+1} = \epsilon_k$. Go to *Step 2.* (iv) Otherwise [unsuccessful], then set $S_{k+1} = S_k$ and $\epsilon_{k+1} = \gamma_d \epsilon_k$, where $0 < \gamma_d < 1$. Go to *Step 2.*

In Procedure 2, optimization problems $(*)$ and $(**)$ are both problems of eigenvalue minimization under LMI constraints which can be solved by using function "gevp" in the LMI Control Toolbox.

Remark 7. In optimization problem $(**)$ of Procedure 2, the following inequalities are equivalent

$$\begin{bmatrix} -\epsilon_k I_n & \Delta S \\ \Delta S^T & -I_n \end{bmatrix} \le 0 \Leftrightarrow \|\Delta S\|^2 \le \epsilon_k$$

by using the Schur Complement [12]. If the size of trust region ϵ_k is too small, the algorithm may miss an opportunity to take a substantial step that will move it much closer to the minimizer of the objective function of $(**)$. If the size of trust region ϵ_k is too large, $M(S, \mathcal{P}) + L(\Delta S, S, \mathcal{P})$ may be far from $M(S + \Delta S, \mathcal{P})$ in the region, so $\bar{\mu}_k$ may be larger than ς_k or μ_k. Therefore, the size of trust region ϵ_k should be changed according to ρ (in *Step 4*(i)), i.e. the performance of the algorithm in previous iterations.

Theorem 2. *By using Procedure* 2, $\mu_{opt2} \le \mu^* = \mu_{k^*} \le \cdots \le \mu_0 \le \mu_{opt1}$, *i.e.* μ^* *is the suboptimal solution.*

Proof (*By Induction*)**.** Since $\mu_0 = \mu_{opt1}$, then $\mu_0 \le \mu_{opt1}$ holds. If the algorithm steps out at the k's iteration, then we assume $\mu_{k-1} \le \cdots \le \mu_0 \le \mu_{opt1}$ holds. Without loss of generality, we assume $\mu_k \le \cdots \le \mu_0 \le \mu_{opt1}$ holds at the k's iteration and the algorithm does not step out. In this case, we need to prove that $\mu_{k+1} \le \mu_k \le \cdots \le \mu_0 \le \mu_{opt1}$ still holds.

$\mu_k = \lambda_{\max}\left[\mathcal{M}\left(S_k, \mathcal{P}_k^*\right)\right]$, otherwise if $\mu_k > \lambda_{\max}\left[\mathcal{M}\left(S_k, \mathcal{P}_k^*\right)\right]$, then it contradicts with the fact that μ_k is the minimizer of the optimization problem $(*)$ in Procedure 2. Otherwise if $\mu_k < \lambda_{\max}\left[\mathcal{M}\left(S_k, \mathcal{P}_k^*\right)\right]$, then it contradicts with the constrain $\mathcal{M}\left(S_k, \mathcal{P}_k^*\right) \le \mu_k I$. Similarly,

$$\varsigma_k = \lambda_{\max}\left[\mathcal{M}\left(S_k, \mathcal{P}_k^*\right) + \mathcal{L}\left(\Delta S_k^*, S_k, \mathcal{P}^*\right)\right].$$

Since $\mathcal{M}\left(S_k, \mathcal{P}_k^*\right) + \mathcal{L}\left(0, S_k, \mathcal{P}_k^*\right) = \mathcal{M}\left(S_k, \mathcal{P}_k^*\right)$, we have

$$\begin{aligned} \varsigma_k &= \lambda_{\max}\left[\mathcal{M}\left(S_k, \mathcal{P}_k^*\right) + \mathcal{L}\left(\Delta S_k^*, S_k, \mathcal{P}^*\right)\right] \\ &\le \lambda_{\max}\left[\mathcal{M}\left(S_k, \mathcal{P}_k^*\right) + \mathcal{L}\left(0, S_k, \mathcal{P}^*\right)\right] \\ &= \lambda_{\max}\left[\mathcal{M}\left(S_k, \mathcal{P}^*\right)\right] = \mu_k. \end{aligned} \tag{33}$$

If $\lvert(\mu_k - \varsigma_k)/(\mu_k + \varsigma_k)\rvert < \varepsilon$ (*Step 4*(i)), then step out (which is not considered in this proof). Otherwise if $\rho > \eta_s > 0$ (*Step 4*(ii) and (iii)), i.e. $(\mu_k - \bar{\mu}_k)/(\mu_k - \varsigma_k) > 0$, then $\bar{\mu}_k < \mu_k$ (since $\varsigma_k \le \mu_k$ by (33) and $\varsigma_k \ne \mu_k$ by $\lvert(\mu_k - \varsigma_k)/(\mu_k + \varsigma_k)\rvert \ge \varepsilon > 0$). In this case, we have

$$\begin{aligned} \mu_{k+1} &= \lambda_{\max}\left[\mathcal{M}\left(S_{k+1}, \mathcal{P}_{k+1}^*\right)\right] \le \lambda_{\max}\left[\mathcal{M}\left(S_{k+1}, \mathcal{P}_k^*\right)\right] \\ &= \lambda_{\max}\left[\mathcal{M}\left(S_k + \Delta S_k^*, \mathcal{P}_k^*\right)\right] = \bar{\mu}_k < \mu_k. \end{aligned}$$

Otherwise (*Step 4*(iv)), we have

$$\begin{aligned}\mu_{k+1} &= \lambda_{\max}\left[\mathcal{M}\left(S_{k+1}, \mathcal{P}_{k+1}^{*}\right)\right] \leq \lambda_{\max}\left[\mathcal{M}\left(S_{k+1}, \mathcal{P}_{k}^{*}\right)\right] \\ &= \lambda_{\max}\left[\mathcal{M}\left(S_{k}, \mathcal{P}_{k}^{*}\right)\right] = \bar{\mu}_k = \mu_k.\end{aligned}$$

Therefore, $\mu_{k+1} \leq \mu_k \leq \cdots \leq \mu_0 \leq \mu_{opt1}$.

Since $\mu_{opt2} \leq \mu_{k+1}$, otherwise it contradicts with the fact that μ_{opt2} is the minimal value of (30). Therefore, $\mu_{opt2} \leq \mu^* = \mu_{k^*} \leq \cdots \leq \mu_0 \leq \mu_{opt1}$, i.e. μ^* is the suboptimal solution. □

Remark 8. This implies that if the original criteria as (22) have feasible solutions, i.e. $\mu_{opt1} < 0$, then the extended criteria as (28) also have feasible solutions which can be obtained through Procedure 2. However, this does not hold vice versa. This will show in Example 2.

Example 2. Consider an uncertain nonlinear neutral system (20) with $D = \begin{bmatrix} 0.3 & 0 \\ 0 & -0.3 \end{bmatrix}$, $A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}$, $A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$, $a = 0.05$, $b = 0.1$, $\tau = 1$. The problem is to determine whether system (20) is exponentially stable with decay rate $\alpha = 0.1$.

There does not exist a feasible solution for criterion (22) proposed in [8], i.e. criterion (28) with $S = 0$. Let $\varepsilon = 0.001$, $\epsilon_0 = 0.1$, $\eta_v = 0.7$, $\eta_i = 0.3$, $\gamma_d = 0.5$, $\gamma_i = 1.5$,

$$\mathcal{D}_{\mathcal{P}_2} = \left\{ \mathcal{P}_2 \left| \begin{array}{lll} 0.1I_2 & \leq & P_1, Q_1, Q_2, Q_3 \leq 100I_2 \\ 0.1 & \leq & \varepsilon_1, \varepsilon_2 \leq 100, \varepsilon_1, \varepsilon_2 \in \mathbb{R} \\ & & P_1, Q_1, Q_2, Q_3 \in \mathbb{R}^{2\times 2} \end{array} \right. \right\}$$

and following Procedure 2 for criterion (28), we obtain $k^* = 2$, $\mu^* = -0.0568$ and

$$S^* = \begin{bmatrix} 0.14 & -0.01 \\ -0.015 & -0.22 \end{bmatrix} \quad P^* = \begin{bmatrix} 46.85 & 0.19 \\ 0.19 & 3.62 \end{bmatrix} \quad Q_1^* = \begin{bmatrix} 80.26 & 1.67 \\ 1.67 & 4.61 \end{bmatrix}$$
$$Q_2^* = \begin{bmatrix} 15.74 & -1.22 \\ -1.22 & 1.46 \end{bmatrix} \quad Q_3^* = \begin{bmatrix} 4.89 & 1.20 \\ 1.20 & 1.79 \end{bmatrix} \quad \varepsilon_1^* = 99.99, \quad \varepsilon_2^* = 66.90.$$

Therefore, system (20) is exponentially stable with decay rate $\alpha = 0.1$.

Remark 9. By selecting appropriate slack matrices through Procedure 1 and Procedure 2, feasible solutions of the extended stability criteria can be obtained for Examples 1 and 2, whereas the original criteria (5) and (22) are not applicable here. The improvement of the proposed model transformation method over its predecessors is demonstrated.

5. Conclusions

By introducing a slack matrix into the equivalent augmented model proposed by Fridman et al., a new model transformation method is proposed in this paper for the stability analysis of a class of neutral type systems. By using the new model transformation method and following the Lyapunov functional approach, a class of existing stability criteria are extended to less conservative ones in terms of nonlinear matrix inequalities. Procedures to solve the nonlinear matrix inequalities that represent the extended criteria are also proposed. Two illustrative examples indicate that the two extended stability criteria have feasible solutions when the corresponding original criteria are not applicable.

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References

[1] S.-I. Niculescu, Delay Effects on Stability: A Robust Control Approach, Springer, Berlin, 2001.
[2] E. Fridman, New Lyapunov–Krasovskii functionals for stability of linear retarded and neutral type systems, Systems Control Lett. 43 (2001) 309–319.
[3] E. Fridman, U. Shaked, Delay-dependent stability and H_∞ control: Constant and time-varying delays, Internat. J. Control 76 (2003) 48–60.
[4] A. Bellen, N. Guglielmi, A.E. Ruehli, Methods for linear systems of circuits delay differential equations of neutral type, IEEE Trans. Circuits Syst. 46 (1999) 212–216.
[5] J.P. Richard, Time-delay systems: An overview of some recent advances and open problems, Automatica 39 (2003) 1667–1694.
[6] P. Agathoklis, S. Foda, Stability and the matrix Lyapunov equation for delay differential systems, Internat. J. Control 49 (1989) 417–432.
[7] E. Rogers, K. Galkowski, D.H. Owens, Control Systems Theory and Applications for Linear Repetitive Processes, Springer, Berlin, 2007.
[8] Y. Chen, A. Xue, R. Lu, S. Zhou, On robustly exponential stability of uncertain neutral systems with time-varying delays and nonlinear perturbations, Nonlinear Anal. 68 (2008) 2464–2470.
[9] Y. He, M. Wu, J.-H. She, G.-P. Liu, Delay-dependent robust stability criteria for uncertain neutral systems with mixed delays, Systems Control Lett. 51 (2004) 57–65.
[10] M. Wu, Y. He, J.-H. She, New delay-dependent stability criteria and stabilizing method for neutral systems, IEEE Trans. Automat. Control 49 (2004) 2266–2271.
[11] J. Nocedal, S.J. Wright, Numerical Optimization, Springer, New York, 1999.
[12] S. Boyd, L.E. Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia, 1994.