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Brief Paper

Linear matrix inequality approach for stability analysis of linear neutral systems in a critical case

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Abstract: This study mainly focuses on the stability of a class of linear neutral systems in a critical case, that is, the spectral radius of the principal neutral term (matrix H) is equal to 1. It is difficult to determine the stability of such systems by using existing methods. In this study, a sufficient stability criterion for the critical case is given in terms of the existence of solutions to a linear matrix inequality (LMI). Moreover, it is also shown that the proposed stability criterion conforms with a fact that the considered linear neutral systems are unstable when H has a Jordan block corresponding to the eigenvalue of modulus 1. An illustrative example is presented to determine the stability of a linear neutral system whose principal neutral term H has multiple eigenvalues of modulus 1 without Jordan chains. This is difficult in existing studies.

1 Introduction

For clarity, we first introduce a class of linear neutral systems

$$\dot{x}(t) - H\dot{x}(t-\tau) = F(x_t) \tag{1}$$

where $\tau > 0$ is a constant delay, $F(\cdot)$ is a linear functional and $x_t \triangleq x(t+\theta)$, $\theta \in [-\tau, 0]$. Based on spectral radius of matrix H, the neutral system (1) can be classified into three cases: $\rho(H) < 1$, $\rho(H) > 1$ and $\rho(H) = 1$. The case $\rho(H) < 1$, namely matrix H is Schur stable, is a necessary condition for exponential stability of the linear neutral system (1) [1, 2]. To the best knowledge of the authors, the case $\rho(H) > 1$ means that there are characteristic roots of the linear neutral system (1) with positive real part, so the system is unstable. The last case $\rho(H) = 1$ is the critical case, which is concerned in this paper.

Neutral systems in the critical case need to be considered in practice because they are in fact related to a class of repetitive control systems [3, 4]. However, it is much more complicated to determine the stability of such systems because their characteristic equation may have an infinite sequence of roots with negative real parts approaching zero. In recent years, stability problem of neutral systems in the critical case is investigated by frequency-domain methods [5, 6] (the interested readers could consult [5] and [6], and references therein, for the development on such a problem). As we know, the frequency-domain stability criteria will become more and more difficult to verify as the dimension of matrix H increases. Moreover, when H has multiple eigenvalues of modulus 1 without Jordan chain, the analysis of non-exponential asymptotic stability is still an 'open problem' [5, pp. 426–427]. The difficulty remains when time-domain methods are used. In most of existing literature, the candidate Lyapunov functionals usually include a non-negative term like $\|D(x_t)\|^2$, where $D(\cdot)$ is called D operator [1, pp. 286–287] and is defined as $D(x_t) = x(t) - Hx(t-\tau)$ for (1). In the case $\rho(H) < 1$. it can be proved that the zero solution of $D(x_t) = 0$ is asymptotically stable when $\|D(x_t)\|^2$ approaches zero asymptotically. However, we cannot obtain the property in the critical case, thus cannot further analyse stability by investigating the tendency of $\|D(x_t)\|$. On the other hand, the other type of stability criteria usually rely on the condition $\rho(H) < 1$ to prove the boundedness of $\|\dot{x}(t)\|$ [7, pp. 336–337; 8, pp. 157–158]. Unfortunately, it is difficult to obtain the boundedness of $\|\dot{x}(t)\|$ in the critical case as well (see the beginning of Section 4.1). Therefore the existing stability criteria cannot cover the critical case easily. In fact, most of existing stability criteria have implicitly assumed $\rho(H) < 1$ [9–12].

In this paper, we mainly investigate the critical case of a class of linear neutral systems. A sufficient delay-independent stability criterion for the critical case is given in terms of the existence of solutions to an LMI. This makes the proposed criterion quite feasible with the aid of a computer. Then, by the proposed criterion, an existing criterion is extended to determine the stability of a scalar linear neutral system in the critical case. Finally, it is shown that the proposed criterion conforms with a fact that the considered linear neutral system is unstable when H has a Jordan block corresponding to the eigenvalue of modulus 1 [5, pp. 394, 415]. An illustrative example shows the effectiveness of the proposed criterion and gives an alternative to handle the 'open problem' according to [5, pp. 426–427].

2 Notation

The notation used in this paper is as follows. $\mathbb{R}^n$ is Euclidean space of dimension n. $\|\cdot\|$ denotes the Euclidean norm or a matrix norm induced by the Euclidean norm. $\mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ denotes the space of continuous n-dimensional vector functions on $[-\tau, 0]$. The symbol $\|\cdot\|_w$ stands for the norm defined by $\|x_t\|_W \triangleq [\|x_t(0)\|^2 + \int_{-\tau}^{0} \|\dot{x}_t(\theta)\|^2 d\theta]^2$, where $x_t \in \mathcal{C}([-\tau, 0]; \mathbb{R}^n)$. $\rho(X)$ and $\lambda_{\min}(X)$ denote the spectral radius and the minimum eigenvalue of matrix X, respectively. X^T and X^* are used for the transpose and conjugate transpose of matrix X. $\mathrm{tr}(X)$ denotes the trace of matrix X. $X > 0$ ($X \geq 0$, $X < 0$, $X \leq 0$) denotes that matrix X is a positive definite (positive semidefinite, negative definite, negative semidefinite) matrix. I_n is the identity matrix with dimension n. '0' denotes a scalar or a zero matrix (vector) of appropriate dimension. '#' in matrices denotes the term which is not used in the development. Sometimes, the dimension of a matrix will not be mentioned when no confusion arises.

3 Problem formulation and preliminary results

For simplicity, we consider a special case of (1) as follows

$$\dot{x}(t) - H\dot{x}(t-\tau) = A_0 x(t) + A_1 x(t-\tau) \tag{2}$$

with the initial condition

$$x(t) = \phi(t), \forall t \in [-\tau, 0]$$

where $x(t) \in \mathbb{R}^n$, $\tau > 0$ is a constant delay and H, A_0, $A_1 \in \mathbb{R}^{n\times n}$ are constant system matrices. $\phi(t)$ is a continuously differentiable smooth vector valued function representing the initial condition function for the interval of $[-\tau, 0]$. The purpose of this paper is to derive a stability criterion in terms of LMIs for the linear neutral system (2) with $\rho(H) \leq 1$, especially for the critical case. In this paper, we do not consider the case of mixed retarded-neutral type systems, that is, when $H \neq 0$, $\det(H) = 0$; and limit ourselves lo one principal neutral term as in [5].

Before proceeding further, we have the following preliminary results (the proofs are all shown in the Appendix):

Lemma 1: For any negative semidefinite matrix $\Phi = \Phi^T \in \mathbb{R}^{n\times n}$, if $\varphi_{kk} = 0$, then $\varphi_{kj} = 0$ and $\varphi_{jk} = 0$, $j = 1, \ldots, n$, where φ_{ij} corresponds to the element in the ith row and jth column of Φ.

Lemma 2: For any $T, H \in \mathbb{R}^{n\times n}$, if H is non-singular and there exist matrices $0 < P = P^T \in \mathbb{R}^{n\times n}$, $0 < Q = Q^T \in \mathbb{R}^{n\times n}$ such that

$$E = \begin{bmatrix} \# & (P+TQ)H \\ \# & H^T QH - Q \end{bmatrix} \leq 0 \tag{3}$$

then $Q > 0$, that is $\lambda_{\min}(Q) > 0$, where $E = E^T$.

Lemma 3: For any given $0 < Q = Q^T \in \mathbb{R}^{n\times n}$, if there exists a matrix $H \in \mathbb{R}^{n\times n}$ such that $H^T QH - Q < 0 (\leq 0)$, then $\rho(H) < 1 (\leq 1)$.

Lemma 4: If there exist matrices $0 \leq Q = Q^T \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times n}$ such that $G^T QG - Q \leq 0$ where $GG^T = I_n$, then $G^T QG - Q = 0$.

Remark 1: Lemma 3 indicates that for any given $Q > 0$, if $\rho(H) = 1$ and the inequality $H^T QH - Q \leq 0$ holds, then $\lambda_{\max}(H^T QH - Q) = 0$. Lemma 3 also implies that if $\rho(H) > 1$, then $H^T QH - Q \leq 0$ does not hold for all $Q > 0$.

4 Main results

In this section, a delay-independent stability criterion (Theorem 1) in terms of an LMI is proposed for the linear neutral system (2) with $\rho(H) \leq 1$. Then, an existing criterion is extended to determine the stability of a scalar linear neutral system in the critical case (Theorem 2). Finally, we prove that the proposed delay-independent stability criterion does not hold when matrix H has a Jordan block corresponding to the eigenvalue of modulus 1 (Theorem 3).

4.1 Stability criterion

The condition $\rho(H) < 1$ usually plays a role to show $\|\dot{x}(t)\|$ being bounded. This is a very important step to show asymptotical stability of neutral type systems [1, pp. 296–297; 7, pp. 330–331, 336–337; 8, pp. 157–158]. If we have obtained that $\|x(t)\|$ is bounded, then

$$\|\dot{x}(t) - H\dot{x}(t-\tau)\| \le (\|A_0\| + \|A_1\|) \sup_{t\in[0,\infty)} \|x(t)\|$$

by (2). Consequently, $\|\dot{x}(t)\|$ is bounded by applying $\rho(H) < 1$. This is not true in the critical case. Taking this into account, we need to seek another condition to replace the boundedness of $\|\dot{x}(t)\|$. To begin with, we need

Definition 1 ([13, p. 123]): Suppose $g(t) : [0, \infty) \to \mathbb{R}$. We say that $g(t)$ is uniformly continuous on $[0, \infty)$ if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|g\ (t+h) - g(t)| < \varepsilon$ for all t on $[0, \infty)$ with $|h| < \delta$.

Barbalat's Lemma ([13, p, 123]): If the differentiate function $f(t)$ has a finite limit as $t \to \infty$, and if $\dot{f}$ is uniformly continuous, then $\dot{f}(t) \to 0$ as $t \to \infty$.

Uniform continuity is often awkward to assert from the definition. A very simple sufficient condition for a differentiable function to be uniformly continuous is that its derivative is bounded. By this condition, many proofs are to show the boundedness of the derivative rather than its uniform continuity, although the latter in fact may play the same role as the former. In the following proof, we will need to show the uniform continuity from the definition.

Before introducing the following Theorem 1, a stability definition is given. It should be noticed that the following definition is slightly different from that proposed in [1]. In [1], the initial condition is restricted by $\sup_{\theta\in[-\tau,0]} \|\phi(\theta)\| < \delta$ rather than $\|\phi\|_W < \delta$. The later depends on the derivative of the initial condition.

Definition 2 ([8, pp. 128, 157]): The trivial solution of the system (2) is said to be stable if for any $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that $\|\phi\|_W < \delta$ implies $\|x(t, \phi)\| < \varepsilon, t \ge 0$. The trivial solution is said to be globally asymptotically stable if it is stable and $\lim_{t\to\infty} \|x(t, \phi)\| = 0$ for any initial condition $\|\phi\|_W < \infty$.

Theorem 1: The solution $x(t, \phi)$ of (2) is globally asymptotically stable, if H is non-singular and there exist matrices $0 < W = W^T \in \mathbb{R}^{n\times n}$, $0 < P = P^T \in \mathbb{R}^{n\times n}$, $0 < Q = Q^T \in \mathbb{R}^{n\times n}$ such that

$$\Omega + LWL^T \le 0 \tag{4}$$

where

$$\Omega = \begin{bmatrix} A_0^T P + PA_0 + S_1^T QS_1 & (P + S_1^T Q)H \\ H^T(P + QS_1) & H^T QH - Q \end{bmatrix},$$

$$L = \begin{bmatrix} I_n & 0 \end{bmatrix}^T \in R^{2n\times n},\ S_1 = A_0 + H^{-1}A_1$$

Proof: The proof is composed of four propositions: Proposition 1 is to show $x(t, \phi) \in \mathcal{L}_\infty[0, \infty)$; Proposition 2 is to show $x(t, \phi) \in \mathcal{L}_2[0, \infty)$; Proposition 3 is to show that $\|x(t, \phi)\|^2$ is uniformly continuous; Proposition 4 is to show the solution $x(t, \phi)$ is stable. If the four propositions are satisfied, then the solution $x(t, \phi)$ of (2) is globally asymptotically stable. The outline of the proof is as follows. Let $f(t) = \int_0^t \|x(s, \phi)\|^2 \,ds$, then $\dot{f}(t) = \|x(t, \phi)\|^2$. Since $\|x(t, \phi)\|^2$ is continuous by Proposition 3, $f(t)$ is a differentiate function. Moreover, $f(t)$ has a finite limit as $t \to 0$ by Proposition 2 and $\dot{f}(t)$ is uniformly continuous by Proposition 3. It follows that $\lim_{t\to\infty} x(t, \phi) = 0$ by Barbaiat's Lemma. Moreover, the solution $x\ (t, \phi)$ is stable by Proposition 4, therefore the solution $x(t, \phi)$ of (2) is asymptotically stable by Definition 2. Next, the four propositions above are proven one by one in detail.

Proposition 1: $x(t, \phi) \in \mathcal{L}_\infty[0, \infty)$.

If H is non-singular, then the neutral system (2) can be rewritten as

$$\begin{aligned} \dot{x}(t) + H^{-1}A_1 x(t) &= H[\dot{x}(t-\tau) + H^{-1}A_1 x(t-\tau)] \\ &\quad + (A_0 + H^{-1}A_1)x(t) \end{aligned}$$

Define $z(t) \triangleq \dot{x}(t) - S_0 x(t)$, then the equation above becomes

$$z(t) = Hz(t-\tau) + S_1 x(t) \tag{5}$$

where $S_0 = -H^{-1}A_1$ and $S_1 = A_0 + H^{-1}A_1$. Choose a candidate Lyapunov-Krasovskill functional to be

$$V(t) = x(t)^T Px(t) + \int_{t-\tau}^{t} z(s)^T Qz(s)\,ds \tag{6}$$

where $0 < P = P^T \in \mathbb{R}^{n\times n}$ and $0 \le Q = Q^T \in \mathbb{R}^{n\times n}$. Note that $\dot{x}(t)$ can be represented as $\dot{x}(t) = S_0 x(t) + z(t)$, then the lime derivative of $V(t)$ is calculated as follows

$$\begin{aligned} \dot{V}(t) &= x(t)^T(S_0^T P + PS_0)x(t) + 2x(t)^T Pz(t) \\ &\quad + z(t)^T Qz(t) - z(t-\tau)^T Qz(t-\tau) \end{aligned}$$

Substituting (5) into the above equation yields

$$\dot{V}(t) = Y(t)^T \Omega Y(t) \tag{7}$$

where $Y(t) = \begin{bmatrix} x(t)^T & z(t-\tau)^T \end{bmatrix}^T$. Since $\Omega \le -LWL^T$ by (4), (7) becomes

$$\begin{aligned} \dot{V}(t) &\le -Y(t)^T LWL^T Y(t) \\ &= -x(t)^T Wx(t) \end{aligned} \tag{8}$$

Since $W > 0$, $\dot{V}(t) \le 0$. It gives $V(t) \le V(0)$. From (6), $x(t)$

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is bounded as

$$\sup_{t\in[0,\infty)} \|x(t)\| \leq b_1 \tag{9}$$

where $b_1 = \sqrt{V(0)/\lambda_{\min}(P)}$. Therefore $x(t, \phi) \in \mathcal{L}_\infty[0, \infty)$.

Proposition 2: $x(t, \phi) \in \mathcal{L}_2[0, \infty)$.

Integrating both sides of (8) from 0 to t, we obtain

$$V(t) \leq V(0) - \lambda_{\min}(W)\int_0^t \|x(s)\|^2\,\mathrm{d}s$$

Since $V(t) \geq 0$ and $\lambda_{\min}(W) > 0$, $\int_0^t \|x(s)\|^2\,\mathrm{d}s \leq V(0)/\lambda_{\min}(W)$. Consequently,

$$\lim_{t\to\infty}\int_0^t \|x(s)\|^2\,\mathrm{d}s \leq V(0)/\lambda_{\min}(W) \tag{10}$$

Therefore $\int_0^t \|x(s)\|^2\,\mathrm{d}s$ has a limit as $t \to \infty$ by (10), that is $x(t) \in \mathcal{L}_2[0, \infty)$

Proposition 3: $\|x(t, \phi)\|^2$ is uniformly continuous.

Since $\phi(t)$ is continuously differentiable, the solution $x(t, \phi)$ is continuously differentiable except maybe at the points $t_0 + k\tau$, $k = 0, 1, 2\ldots$ [1, p. 25, Theorem 7.1]. Then, by Newton–Leibniz Formula, we have

$$x(t+h) - x(t) = \int_t^{t+h} \dot{x}(s)\,\mathrm{d}s$$

where $h > 0$ without loss of generality. Utilising (9) and $\dot{x}(t) = S_0x(t) + z(t)$, we have

$$\begin{aligned}|\|x(t+h)\|^2 - \|x(t)\|^2| &\leq 2b_1\|x(t+h) - x(t)\|\\ &\leq 2b_1\int_t^{t+h}\|\dot{x}(s)\|\,\mathrm{d}s\\ &= 2b_1\int_t^{t+h}\|S_0x(s) + z(s)\|\,\mathrm{d}s\\ &\leq 2b_1\left(b_1\|S_0\|h + \int_t^{t+h}\|z(s)\|\,\mathrm{d}s\right)\end{aligned} \tag{11}$$

Using the Cauchy–Schwarz inequality $\langle a, b\rangle \leq \langle a, a\rangle^{1/2}\langle b, b\rangle^{1/2}$ we obtain

$$\int_t^{t+h}\|z(s)\|\,\mathrm{d}s \leq \left(\int_t^{t+h} 1^2\,\mathrm{d}s\right)^{1/2}\left[\int_t^{t+h}\|z(s)\|^2\,\mathrm{d}s\right]^{1/2} \tag{12}$$

Since $\Omega + LWL^T \leq 0$, $\lambda_{\min}(Q) > 0$ by Lemma 2. Then noticing (6), we obtain

$$\sup_{t\in[0,\infty)}\int_{t-\tau}^t \|z(s)\|^2\,\mathrm{d}s \leq b_2$$

where $b_2 = V(0)/\lambda_{\min}(Q)$. Thus (12) becomes

$$\int_t^{t+h}\|z(s)\|\,\mathrm{d}s \leq N\sqrt{b_2}\sqrt{h}$$

where $N = \lfloor h/\tau\rfloor + 1$, $\lfloor h/\tau\rfloor$ represents the nearest integer of h/τ. Therefore inequality (11) becomes

$$|\|x(t+h)\|^2 - \|x(t)\|^2| \leq 2b_1(b_1\|S_0\|h + N\sqrt{b_2}\sqrt{h})$$

This implies that $|x(t, \phi)|^2$ is uniformly continuous. □

Proposition 4: $x(t, \phi)$ is stable.

Since $\phi(t)$ is continuously differentiable, the solution $x(t, \phi)$ is continuously differentiable except maybe at the points $k\tau$, $k = 0, 1, 2, \ldots$ [1, p. 25, Theorem 7.1]. Then, by Newton-Leibniz Formula, it follows that

$$x(s) = x(t) - \int_s^t \dot{x}(\zeta)\mathrm{d}\zeta$$

for $s \in [t-\tau, t]$. Based on the equation above, we have

$$\begin{aligned}\int_{t-\tau}^t \|x(s)\|^2\mathrm{d}s &= \int_{t-\tau}^t\left\|x(t) - \int_s^t\dot{x}(\zeta)\mathrm{d}\zeta\right\|^2\mathrm{d}s\\ &\leq 2\tau\|x(t)\|^2 + 2\int_{t-\tau}^t\left\|\int_s^t\dot{x}(\zeta)\mathrm{d}\zeta\right\|^2\mathrm{d}s\\ &\leq 2\tau\|x(t)\|^2 + 2\int_{t-\tau}^t\left(\int_s^t\|\dot{x}(\zeta)\mathrm{d}\zeta\right)^2\mathrm{d}s\end{aligned}$$

Using the Cauchy–Schwarz inequality $\langle a, b\rangle^2 \leq \langle a, a\rangle\langle b, b\rangle$, we obtain

$$\left(\int_s^t\|\dot{x}(\zeta)\mathrm{d}\zeta\right)^2 \leq (t-s)\int_s^t\|\dot{x}(\zeta)\|^2\mathrm{d}\zeta \leq \tau\int_{t-\tau}^t\|\dot{x}(\zeta)\|^2\mathrm{d}\zeta$$

Consequently, we have

$$\int_{t-\tau}^t\|x(s)\|^2\mathrm{d}s \leq 2\tau\|x(t)\|^2 + 2\tau^2\int_{t-\tau}^t\|\dot{x}(s)\|^2\mathrm{d}s.$$

By using the inequality above, $V(t)$ is bounded as

$$\begin{aligned}V(t) &= \lambda_{\max}(P)\|x(t)\|^2 + \lambda_{\max}(Q)\int_{t-\tau}^t\|\dot{x}(s) - S_0x(s)\|^2\mathrm{d}s\\ &\leq \lambda_{\max}(P)\|x(t)\|^2 + 2\lambda_{\max}(Q)\int_{t-\tau}^t\|\dot{x}(s)\|^2\mathrm{d}s\\ &\quad + 2\lambda_{\max}(Q)\|S_0\|^2\int_{t-\tau}^t\|x(s)\|^2\mathrm{d}s\end{aligned}$$

Substituting the result $\int_{t-\tau}^{t}\|x(s)\|^2\mathrm{d}s \le 2\tau\|x(t)\|^2 + 2\tau^2\int_{t-\tau}^{t}\|\dot{x}(s)\|^2\mathrm{d}s$ into the inequality above results in

$$V(t) \le \rho_1\|x(t)\|^2 + \rho_2\int_{t-\tau}^{t}\|\dot{x}(s)\|^2\mathrm{d}s$$
$$\le \max(\rho_1, \rho_2)\|x_t\|_W^2$$

where $\rho_1 = \lambda_{\max}(P) + 4\tau\lambda_{\max}(Q)\|S_0\|^2$ and $\rho_2 = 2\lambda_{\max}(Q) + 4\tau^2\lambda_{\max}(Q)\|S_0\|^2$. Therefore, $V(0) \le \max(\rho_1, \rho_2)\|\phi\|_W^2$. For any $\varepsilon > 0$, there exists a $\delta(\varepsilon) = \varepsilon\sqrt{\lambda_{\min}(P)/\max(\rho_1, \rho_2)} > 0$ such that $\|\phi\|_W < \delta(\varepsilon)$ implies that the inequality (9) becomes

$$\sup_{t\in[0,\infty]}\|x(t)\| \le \sqrt{V(0)/\lambda_{\min}(P)}$$
$$\le \sqrt{\max(\rho_1, \rho_2)\|\phi\|_W^2/\lambda_{\min}(P)}$$
$$\le \sqrt{\max(\rho_1, \rho_2)\delta(\varepsilon)^2/\lambda_{\min}(P)}$$
$$= \varepsilon$$

Therefore, the solution $x(t, \phi)$ is stable by Definition 2.

Remark 2: If $\Omega + LWL^T < 0$, then $\Omega < 0$. As a result, we obtain $H^TQH - Q < 0$. This implies $\rho(H) < 1$ by Lemma 3. Therefore if $\rho(H) = 1$, then the matrix inequality (4) must have the form $\Omega + LWL^T \le 0$ rather than $\Omega + LWL^T < 0$. When the conditions of Theorem 1 are satisfied, the solution $x(t, \phi)$ of (2) is exponentially stable in the case with $\rho(H) < 1$ [2], whereas the solution $x(t, \phi)$, non-exponentially stable in the critical case [5, p. 413].

Remark 3: In Theorem 1, the condition $Q \ge 0$ can be changed to $Q > 0$ by Lemma 2.

Remark 4: If $\rho(H) > 1$, then Theorem 1 does not hold by Lemma 3 (or refer to Remark 1).

4.2 Scalar case

Now, let us consider a scalar linear neutral system

$$\dot{x}(t) - h\dot{x}(t-\tau) = a_0x(t) + a_1x(t-\tau) \tag{13}$$

where $h, a_0, a_1 \in \mathbb{R}$.

Verriest and Niculescu gave the following result:

Lemma 5 ([14]): The scalar neutral system (13) is delay-independent asymptotically stable if (i) $a_0 < 0$, (ii) $|h| < 1$, (iii) $|a_1| < |a_0|$.

By Theorem 1 the extension of Lemma 5 for the critical case is given as follows

Theorem 2: The scalar neutral system (13) is delay-independent asymptotically stable if (i) $a_0 < 0$, (ii) $|h| \le 1$, (iii) $|a_1| < |a_0|$

Proof: When $|h| = 1$, (4) can be written as

$$\begin{bmatrix} 2a_0p + (a_0 + h^{-1}a_1)^2q + w & [p + (a_0 + h^{-1}a_1)q]h \\ [p + (a_0 + h^{-1}a_1)q]h & 0 \end{bmatrix} \le 0 \tag{14}$$

where p, q, $w \in \mathbb{R}$ all positive numbers. Thus, if the following condition

$$\begin{cases} 2a_0p + (a_0 + h^{-1}a_1)^2q < 0 \\ p + (a_0 + h^{-1}a_1)q = 0 \end{cases} \tag{15}$$

holds, then (14) holds with a sufficiently small positive number w. This implies that the scalar linear neutral system (13) is asymptotically stable with $h^2 = 1$ Solving (15) yields $a_0 < 0$ and $|a_1| < |a_0|$.

Combining the above results and Lemma 5, we can conclude this proof. □

Remark 5: When $|h| = 1$, the characteristic equation of system (13) has an infinite sequence of roots with negative real parts approaching zero. As a result, it is difficult to determine the stability of system (13) with $|h| = 1$ by using frequency-domain methods.

4.3 Special case

For simplicity, let $\sigma_1 = \{\lambda \| \lambda| = 1, \lambda \in \lambda(H)\}$. The linear neutral system (2) is unstable when H has a Jordan block corresponding to $\lambda \in \sigma_1$[5, pp. 394, 415]. In this section, we will show that Theorem 1 does not hold in the case.

The Jordan blocks corresponding to $\lambda \in \sigma_1$ have two forms as follows [15, pp. 82–83]

$$D_r = \begin{bmatrix} 1 & 1 & & 0 \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{bmatrix} \in \mathbb{R}^{r\times r} \tag{16}$$

or

$$D_r = \begin{bmatrix} C(\alpha) & I_2 & & 0 \\ & C(\alpha) & \ddots & \\ & & \ddots & I_2 \\ 0 & & & C(\alpha) \end{bmatrix} \in \mathbb{R}^{2r\times 2r},$$
$$C(\alpha) \triangleq \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \tag{17}$$

Lemma 6: If $D_r^TQ_rD_r - Q_r \le 0$ and $0 \le Q_r = Q_r^T$, then Q_r is singular.

Proof: See in Appendix.

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Theorem 3: If matrix H has a Jordan block corresponding to $\lambda \in \sigma_1$, then Theorem 1 does not hold.

Proof: The key point of this proof is to show that Theorem 1 holds with $Q \geq 0$ rather than $Q > 0$ when matrix H has a Jordan block corresponding to $\lambda \in \sigma_1$. But this is a contradiction by Lemma 2.

Suppose, to the contrary, that Theorem 1 holds when matrix H has a Jordan block corresponding to $\lambda \in \sigma_1$. Then $H^T QH - Q \leq 0$ is satisfied. If matrix H has a Jordan block corresponding to $\lambda \in \sigma_1$, then H can be transformed into the real Jordan canonical form [15, p. 83]

$$S_J^{-1} H S_J = H_J \tag{18}$$

where $H_J = \begin{bmatrix} J_o & 0 \\ 0 & D_r \end{bmatrix}$ and D_r has the form as in (16) or (17). Pre-multiplying and post-multiplying $H^T QH - Q \leq 0$ by S_J^T and S_J respectively, we obtain

$$\begin{aligned} &H_J^T Q_J H_J - Q_J \\ &= \begin{bmatrix} J_o^T Q_{J,11} J_o - Q_{J,11} & J_o^T Q_{J,12} D_r - Q_{J,12} \\ D_r^T Q_{J,12}^T J_o - Q_{J,12}^T & D_r^T Q_{J,22} D_r - Q_{J,22} \end{bmatrix} \leq 0 \end{aligned} \tag{19}$$

where $Q_J = S_J^T Q S_J$ and $Q_J = \begin{bmatrix} Q_{J,11} & Q_{J,12} \\ Q_{J,12}^T & Q_{J,22} \end{bmatrix}$. Since $D_r^T Q_{J,22} D_r - Q_{J,22} \leq 0$ by (19), $Q_{J,22}$ is singular by Lemma 5, consequently, Q_J is singular. This implies that Q is singular, that is, Theorem 1 holds with $Q \geq 0$ rather than $Q > 0$. □

Remark 6: By Theorem 3, Theorem 1 conforms with the fact that the system (2) is unstable when H has a Jordan block corresponding to $\lambda \in \sigma_1$ [5, pp. 394, 415].

5 Illustrative example

Consider the linear neutral system (2) in the critical case with

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_0 \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} 0.4 & 0.1 \\ 0.4 & 0.1 \end{bmatrix}$$

In this example, all eigenvalues of H are 1. By the stability criterion (4), we obtain the following solution

$$P = \begin{bmatrix} 3.5025 & -2.1255 \\ -2.1255 & 5.9123 \end{bmatrix}, Q = \begin{bmatrix} 2.3021 & 0.4519 \\ 0.4519 & 7.1216 \end{bmatrix}, \\ W = 0.1 I_2$$

The eigenvalues of $\Omega + LWL^T$ are $(-12.0220, -3.7504, 0, 0)$. Therefore the system considered in this example is asymptotically stable independent of delay.

Remark 7: H in the system considered in the example has multiple eigenvalues of modulus 1 without Jordan chains. The stability analysis of such a system is still an 'open problem' according to [5, pp. 426–427]. However, the stability of the system can be determined by the stability criterion (4).

6 Conclusions

Asymptotic stability of neutral type systems, especially in the critical case, is studied and a stability criterion in terms of LMIs is proposed. It is also shown that the proposed stability criterion conforms with the fact that the considered linear neutral system is unstable when H has a Jordan block corresponding to the eigenvalue of modulus 1. Furthermore, the proposed criterion can help to determine the stability of the case where H has multiple eigenvalues of modulus 1 without Jordan chains. This gives an alternative to handle the 'open problem' according to [5, pp. 426–427].

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9 Appendix

9.1 Proof of Lemma 1

Without loss of generality, take

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{12}^T & \varphi_{kk} & \Phi_{23} \\ \Phi_{13}^T & \Phi_{23}^T & \Phi_{33} \end{bmatrix}, \quad \varphi_{kk} = 0,$$

$$\Phi_{12} = \begin{bmatrix} \varphi_{1k} & \cdots & \varphi_{(k-1)k} \end{bmatrix}^T,$$

$$\Phi_{23} = \begin{bmatrix} \varphi_{k(k+1)} & \cdots & \varphi_{kn} \end{bmatrix}$$

for example, where Φ_{11}, Φ_{13}, Φ_{33} are matrices with appropriate dimensions. Assume $\Phi_{12} \neq 0$ to the contrary that there exists a unitary matrix U with an appropriate dimension such that

$$U^T\Phi_{12}\Phi_{12}^T U = \Lambda \tag{20}$$

where $\Lambda \geq 0$ and $\Lambda \neq 0$. So we can choose $\xi \neq 0$ to satisfy $\xi^T\Lambda\xi > 0$.

Choosing $v = \begin{bmatrix} (U\xi)^T & \mu\Phi_{12}^T U\xi & 0 \end{bmatrix}^T$, we have

$$v^T\Phi v = \xi^T U^T\Phi_{11}U\xi + 2\mu\xi^T U^T\Phi_{12}\Phi_{12}^T U\xi \tag{21}$$

where μ is a scalar and $v \neq 0$. By using (20), the equation above becomes

$$v^T\Phi v = \xi^T U^T\Phi_{11}U\xi + 2\mu\xi^T\Lambda\xi$$

Since $\xi^T\Lambda\xi > 0$, we can choose $\mu > \xi^T U^T\Phi_{11}U\xi / -2\xi^T\Lambda\xi$ to make $v^T\Phi v > 0$. This contradicts with the fact $\Phi \leq 0$. Therefore $\Phi_{12} = 0$. Using the similar method, we can also prove $\Phi_{23} = 0$. □

9.2 Proof of Lemma 2

Suppose, to the contrary, that $\lambda_{\min}(Q) = 0$. Then there exist two cases: $Q = 0$ and $Q \neq 0$. If $Q = 0$, then (3) becomes

$$E = \begin{bmatrix} \# & PH \\ \# & 0 \end{bmatrix} \leq 0.$$

Consequently, $PH = 0$ by Lemma 1. Since $0 < P = P^T$, we obtain $H = 0$. This contradicts with nonsingularity of H. Therefore the remainder of proof only needs to consider $Q \neq 0$. For $Q \neq 0$, there exists a unitary matrix $U \in \mathbb{R}^{n\times n}$ such that

$$UQU^T = \Lambda \tag{22}$$

where $\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$ with $\Lambda_1 > 0$. Pre-multiplying and post-multiplying (3) by Ξ and Ξ^T respectively yields

$$\Xi E \Xi^T = \begin{bmatrix} \# & U(P+TQ)HU^T \\ \# & UH^TQHU^T - UQU^T \end{bmatrix} \leq 0$$

where $\Xi = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$. Furthermore, in light of (22) and the fact $U^TU = I_n$, we have

$$\begin{aligned} \Xi E \Xi^T &= \begin{bmatrix} \# & U(P+TU^TUQ)U^TUHU^T \\ \# & UH^TU^TUQU^TUHU^T - UQU^T \end{bmatrix} \\ &= \begin{bmatrix} \# & (\tilde{P}+\tilde{T}\Lambda)\tilde{H} \\ \# & \tilde{H}^T\Lambda\tilde{H} - \Lambda \end{bmatrix} \leq 0 \end{aligned} \tag{23}$$

where $\tilde{H} = UHU^T$, $\tilde{P} = UPU^T$ and $\tilde{T} = UTU^T$.

By rewriting $\tilde{H}$ as $\tilde{H} = \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{21} & \tilde{H}_{22} \end{bmatrix}$, the term $\tilde{H}^T\Lambda\tilde{H} - \Lambda$ becomes

$$\tilde{H}^T\Lambda\tilde{H} - \Lambda = \begin{bmatrix} \# & \# \\ \# & \tilde{H}_{12}^T\Lambda_1\tilde{H}_{12} \end{bmatrix} \tag{24}$$

Since $\tilde{H}^T\Lambda\tilde{H} - \Lambda \leq 0$ by (23), $\tilde{H}_{12}^T\Lambda_1\tilde{H}_{12} \leq 0$. On the other hand, hence $\tilde{H}_{12}^T\Lambda_1\tilde{H}_{12} \geq 0$ by $\Lambda_1 > 0$, hence

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$\tilde{H}_{12} = 0$. In this case, $(\tilde{P} + \tilde{T}\Lambda)\tilde{H}$ can be written as

$$(\tilde{P} + \tilde{T}\Lambda)\tilde{H} = \begin{bmatrix} \# & \# \\ \# & \tilde{P}_{22}\tilde{H}_{22} \end{bmatrix} \quad (25)$$

where $\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{12}^T & \tilde{P}_{22} \end{bmatrix}$. Substituting (24) and (25) into (23) yields

$$\Xi E \Xi^T = \begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & \tilde{P}_{22}\tilde{H}_{22} \\ \# & \# & \# & \# \\ \# & \# & \# & 0 \end{bmatrix} \leq 0$$

This inequality implies $\tilde{P}_{22}\tilde{H}_{22} = 0$ by Lemma 1. If $\tilde{P}_{22}$ is non-singular, then $\tilde{H}_{22} = 0$, which implies $\tilde{H} = \begin{bmatrix} \tilde{H}_{11} & 0 \\ \tilde{H}_{21} & 0 \end{bmatrix}$. This contradicts with the nonsingularity of H. If $\tilde{P}_{22}$ is singular, then it contradicts with $P > 0$. Therefore $Q > 0$. □

9.3 Proof of Lemma 3

Suppose, to the contrary, that for any given $Q > 0$ there exists a matrix H such that $\rho(H) \geq 1$ and $H^TQH - Q < 0$. Use λ_H to denote an eigenvalue of H where $|\lambda_H| = \rho(H)$, then there exists an eigenvector $v_H \neq 0$ such that $Hv_H = \lambda_H v_H$. Since $H^TQH - Q < 0$, we have $v_H^*(H^TQH - Q)v_H < 0$. Consequently

$$[\rho(H)^2 - 1]v_H^* Q v_H < 0 \quad (26)$$

If $\rho(H) = 1$, then the inequality (26) becomes $0 < 0$ which is a contradiction; On the other hand, if $\rho(H) > 1$, then (26) becomes $v_H^* Q v_H < 0$ which contradicts with $Q > 0$. Therefore, if there exists a matrix H such that $H^TQH - Q < 0$, then $\rho(H) < 1$. Similarly, we can also prove that for any given $Q > 0$. If there exists a matrix H such that $H^TQH - Q \leq 0$, then $\rho(H) \leq 1$. □

9.4 Proof of Lemma 4

Since

$$\begin{aligned} \mathrm{tr}(G^TQG - Q) &= \mathrm{tr}(G^TQG) - \mathrm{tr}(Q) \\ &= \mathrm{tr}(QGG^T) - \mathrm{tr}(Q) \\ &= \mathrm{tr}(QGG^T - Q) \end{aligned}$$

and $GG^T = I_n$, we obtain that

$$\mathrm{tr}(G^TQG - Q) = 0 \quad (27)$$

The equation (27) implies that the sum of the elements on the main diagonal of $G^TQG - Q$ is zero. Moreover, since $G^TQG - Q \leq 0$, every main diagonal element of $G^TQG - Q$ is smaller than or equal to zero. Therefore we can conclude that every diagonal clement of $G^TQG - Q$ is zero. Furthermore, according to *Lemma 1*, we have $G^TQG - Q = 0$. □

9.5 Proof of Lemma 6

D_r in (16) or (17) has a recursive form as $D_k = \begin{bmatrix} D_{k-1} & \# \\ 0 & \# \end{bmatrix}$, $k = 2, \ldots, r$. So $D_k^TQ_kD_k - Q_k$ has the form as $D_k^TQ_kD_k - Q_k = \begin{bmatrix} D_{k-1}^TQ_{k-1}D_{k-1} - Q_{k-1} & \# \\ \# & \# \end{bmatrix}$, where $Q_k = \begin{bmatrix} Q_{k-1} & \# \\ \# & \# \end{bmatrix}$, $k = 2, \ldots, r$. Therefore $D_r^TQ_rD_r - Q_r \leq 0$ implies that $D_2^TQ_2D_2 - Q_2 \leq 0$. If Q_2 is singular, then we can conclude Q_r is singular. The remainder of the proof is to show that Q_2 is singular.

(i) If $D_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $D_2^TQ_2D_2 - Q_2 \leq 0$ can be represented by

$$D_2^TQ_2D_2 - Q_2 = \begin{bmatrix} 0 & q_{11} \\ q_{11} & q_{11} + 2q_{12} \end{bmatrix} \leq 0 \quad (28)$$

where $Q_2 = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}$, $q_{11}, q_{12}, q_{22} \in \mathbb{R}$. If (28) holds, then $q_{11} = 0$ by Lemma 1. This implies that Q_2 singular.

(ii) If $D_2 = \begin{bmatrix} C(\alpha) & I_2 \\ 0 & C(\alpha) \end{bmatrix}$, then

$$D_2^TQ_2D_2 - Q_2 = \begin{bmatrix} d_1 & d_2 \\ d_2^T & \# \end{bmatrix} \leq 0$$

where $Q_2 = \begin{bmatrix} q_{11} & q_{12} \\ q_{12}^T & q_{22} \end{bmatrix}$, $q_{11}, q_{12}, q_{22} \in \mathbb{R}^{2\times 2}$, $d_1 = C(\alpha)^T q_{11}C(\alpha) - q_{11}$ and $d_2 = C(\alpha)^Tq_{11} + C(\alpha)^Tq_{12}\ C(\alpha) - q_{12}$. The above inequality implies $d_1 \leq 0$. Since $C(\alpha)C(\alpha)^T = I_2$ and $q_{11} \geq 0$, $d_1 = 0$ by Lemma 4. Consequently, $d_2 = 0$ by Lemma 1, that is

$$C(\alpha)^Tq_{11} + C(\alpha)^Tq_{12}C(\alpha) - q_{12} = 0$$

Pre-multiplying $C(\alpha)$ on both sides of the above equation and using $C(\alpha)C(\alpha)^T = I_2$, we have

$$q_{11} = -q_{12}C(\alpha) + C(\alpha)q_{12}$$

Then

$$\begin{aligned} \mathrm{tr}(q_{11}) &= -\mathrm{tr}[q_{12}C(\alpha)] + \mathrm{tr}[C(\alpha)q_{12}] \\ &= -\mathrm{tr}[q_{12}C(\alpha)] + \mathrm{tr}[q_{12}C(\alpha)] \\ &= 0 \end{aligned} \quad (29)$$

Since $q_{11} \geq 0$, every diagonal element of q_{11} is larger than or equal to zero. Consequently, similar to the proof of Lemma 4, we obtain $q_{11} = 0$ by (29), This implies that Q_2 is singular. □

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