However, if $D_1 = D_2 = D_3 = 0$, then

$$e^{A_3}e^{3A_2}\langle A_1|B_1\rangle + e^{A_3}\langle A_2|B_2\rangle + \langle A_3|B_3\rangle = \mathrm{span}\{[3e\ 1\ 1\ 0]^T, [0\ 0\ 1\ 0]^T, [0\ 0\ 0\ 1]^T\}.$$

By Corollary 1 in [15] or Corollary 2 in [7], in this case, system (1) is not controllable to origin on $[t_0, t_f]$. It implies that in this example, the delayed terms in system (1) can make the uncontrollable system without these terms controllable.

V. Conclusion

In this note, the issue on the controllability criteria for a class of piecewise linear time-varying impulsive systems with delayed input has been addressed. Several sufficient and necessary conditions for controllability of such systems have been established by variation of parameters. Geometric type conditions for piecewise linear impulsive delayed systems are also discussed. Two examples have been given to show the effectiveness of the proposed results. Furthermore, it is illustrated that the time delay τ is useful to achieve the controllability for the impulsive system. It is worth noticing that this note considers the controllability for a class of impulsive delay systems. Many issues are still untouched on more general systems, for example, the nonlinear systems with more complex impulses.

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A Filtered Repetitive Controller for a Class of Nonlinear Systems

Quan Quan and Kai-Yuan Cai

***Abstract*—In this note, a filtered repetitive controller (FRC) is designed to compensate for periodic disturbances in a class of nonlinear systems. First, a new model of periodic disturbances is proposed. By this model, the FRC is designed and the resulting closed-loop error dynamics are analyzed with the help of a Lyapunov-Krasovskii functional. Finally the method is applied to periodic disturbance rejection in a class of robotic manipulators. Compared with repetitive controllers, the FRC provides the flexibility to choose filter parameters to achieve a tradeoff between tracking performance and stability. More importantly, FRC can deal with small input delay while the corresponding RC cannot.**

***Index Terms*—Additive decomposition, nonlinear systems, repetitive control (RC), robotic manipulator.**

I. Introduction

Repetitive control (RC) is an internal-model-based control approach in which the infinite-dimensional internal model $1/(1 - e^{-sT})$ gives rise to an infinite number of poles on the imaginary axis. It was proved in [1] that, for a class of general linear plants, exponential stability of RC systems could be achieved only when the plant is proper but not strictly proper. Moreover, the internal model $1/(1 - e^{-sT})$ may destabilize the system. To enhance stability, a suitable filter is introduced as

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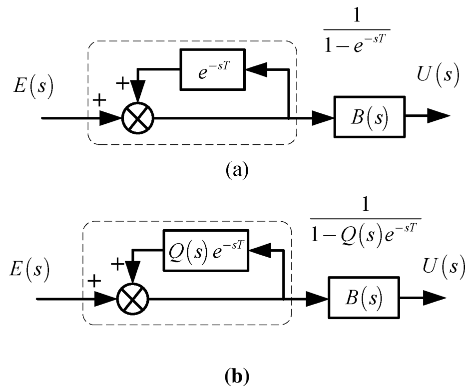

Fig. 1. Suitable filter $Q(s)$ is introduced into a repetitive controller to form an FRC. (a) Repetitive controller. (b) Filtered repetitive controller.

shown in Fig. 1, forming a *filtered repetitive controller* [1] (FRC, or *filtered repetitive control*, also designated FRC) in which the loop gain is reduced at high frequencies. Stability results only with some sacrifice of high frequency performance. With appropriate design, however, an FRC can often achieve an acceptable tradeoff between tracking performance and stability, a tradeoff which broadens the application of RC in practice.

In fact, RC can also be considered as a special case of FRC with an all-pass filter. In the past two decades or more, FRC for linear time-invariant systems has reached maturity [2]–[4]. There has been little research, however, on RC, let alone FRC, for nonlinear systems. This is the initial objective of this note. A linear RC system is a neutral type system in a critical case [5], [6]. The characteristic equation of the neutral type system has an infinite sequence of roots with negative real parts approaching zero, i.e., $\sup\{\mathrm{Re}\ s\,|F(s)=0\}=0$ where $F(s)$ is the characteristic equation. This implies that a sufficiently small uncertainty may lead to $\sup\{\mathrm{Re}\ s\,|F(s)=0\}>0$. It is proved that a linear RC system will lose its stability when subject to an input delay no matter how small the delay is (see *Appendix A*). Therefore, the stability of RC systems is insufficiently robust. On the other hand, with a low-pass filter, an FRC system is a retarded type system. For a stable FRC system, we have $\sup\{\mathrm{Re}\ s\,|F(s)=0\}<0$. Continuity arguments would suggest that for a sufficiently small uncertainty, the FRC system is still stable. Therefore, it is expected that, with a low-pass filter, a nonlinear FRC system with a small input delay is still stable. This is our second objective.

To achieve the two objectives we must analyze the tracking performance of a closed-loop system composed of an FRC and a nonlinear plant. The theory of FRC proposed in [1] is derived in the frequency domain and can be applied only with difficulty, if at all, to nonlinear systems. For this reason, tracking performance needs to be analyzed in the time domain. The contributions of this note are: 1) a method to design and analyze an FRC for a class of nonlinear systems; 2) the ability of our FRC to cope with small input delay; and 3) a tradeoff achieved by tuning filter parameters between tracking performance and stability.

We use the following notation. $\mathbb{R}^n$ is Euclidean space of dimension n and $\mathbb{R}^+$ denotes the space of nonnegative reals in $\mathbb{R}$. $\|\cdot\|$ denotes the Euclidean vector norm or induced matrix norm. $\mathcal{C}([-T,0];\mathbb{R}^n)$ denotes the space of continuous n-dimensional vector functions on $[-T,0]$. $\|x_t\|_c \triangleq \sup_{\theta\in[-T,0]}\|x(t+\theta)\|$, where $x_t \triangleq x_t(\theta)=x(t+\theta), \theta\in[-T,0]$. $\mathcal{C}^n_{PT}([0,\infty);\mathbb{R}^m)$ is the space of nth-order continuously differentiable functions $g:[0,\infty)\rightarrow\mathbb{R}^m$ which are T-periodic, i.e., $g(t+T)=g(t)$. $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote the minimum and maximum eigenvalues respectively of a positive semidefinite matrix X. I_n is the identity matrix with dimension n.

[1]In this note we have replaced the term "modified" in [1] with the more descriptive term "filtered".

II. Problem Formulation and a New Model of Periodic Signals

A. Problem Formulation

To illustrate the generality of FRC, we consider the following error dynamics examined in [7], [8]:

$$\dot{e}(t)=f(t,e(t))+b(t,e(t))\,[v(t)-\hat{v}(t)]. \tag{1}$$

Here $e(t)\in\mathbb{R}^n$ is a tracking error vector, $v(t)\in\mathbb{R}^m$ is a disturbance, $\hat{v}(t)\in\mathbb{R}^m$ is a signal designed to compensate for $v(t)$. For simplicity, we set the initial time $t_0=0$. Throughout this note, we assume the following for the system (1).

Assumption 1 [7], [8]: The functions $f:[0,\infty)\times\mathbb{R}^n\rightarrow\mathbb{R}^n$ and $b:[0,\infty)\times\mathbb{R}^n\rightarrow\mathbb{R}^{n\times m}$ are bounded when $e(t)$ is bounded on $\mathbb{R}^+$. Moreover, there exists a differentiable function $V_0:[0,\infty)\times\mathbb{R}^n\rightarrow[0,\infty)$, a positive real c_0, a positive definite matrix $M(t)=M^T(t)\in\mathbb{R}^{n\times n}$ with $0<\lambda_M I_n<M(t)$ and a matrix $R(t)\in\mathbb{R}^{n\times m}$ such that

$$V_0(t,e(t))\geq c_0\|e(t)\|^2 \tag{2}$$
$$\dot{V}_0(t,e(t))\leq -e^T(t)M(t)e(t)+e^T(t)R(t)[v(t)-\hat{v}(t)]. \tag{3}$$

Assumption 2: The disturbance v satisfies $v\in\mathcal{C}^0_{PT}([0,\infty);\mathbb{R}^m)$.

Under *Assumptions 1–2*, our objective is to design a single FRC with the following two properties: 1) with certain filter parameters, $\lim_{t\rightarrow\infty}e(t)=0$; 2) with another set of appropriate filter parameters, for any value of $e(0)$, $e(t)$ is uniformly ultimately bounded (for the definition see *Definition 1* below), and the FRC can cope with small input delay.

B. A New Model of Periodic Signals

A new model to describe periodic signals is proposed. In order to show the difference, the usual model of periodic signals is given first. Any $v\in\mathcal{C}^0_{PT}([0,\infty);\mathbb{R}^m)$ can be generated by the model [1]

$$\begin{aligned}x(t)&=x(t-T)\\ v(t)&=x(t)\\ x(\theta)&=v(T+\theta),\quad \theta\in[-T,0]\end{aligned} \tag{4}$$

where $t\in\mathbb{R}^+$ and $x(t)\in\mathbb{R}^m$ is the state. By the internal model principle [9], it is expected that asymptotic rejection of a periodic disturbance can be achieved by incorporating the above model (4), i.e., $(1/(1-e^{-sT}))I_m$ (the transfer function of (4)), into the closed-loop system. This is also the basic idea of RC [1].

A new model to describe $v\in\mathcal{C}^0_{PT}([0,\infty);\mathbb{R}^m)$, which will help in design of the FRC for the nonlinear system (1), is given in *Lemma 1*.

Lemma 1: If $A_{0,\epsilon}>0$ and $\|A_{1,\epsilon}\|<1$, then for any $v\in\mathcal{C}^0_{PT}([0,\infty);\mathbb{R}^m)$ there exists a signal $\delta\in\mathcal{C}^0_{PT}([0,\infty);\mathbb{R}^m)$ such that

$$\begin{aligned}A_{0,\epsilon}\dot{x}_p(t)&=-x_p(t)+A_{1,\epsilon}x_p(t-T)+\delta(t)\\ v(t)&=x_p(t)+\delta(t)\\ x_p(\theta)&=v(T+\theta),\quad \theta\in[-T,0]\end{aligned} \tag{5}$$

where $A_{0,\epsilon},A_{1,\epsilon}\in\mathbb{R}^{m\times m}$. If $A_{0,\epsilon}=0$ and $A_{1,\epsilon}=I_m$, then any $v\in\mathcal{C}^0_{PT}([0,\infty);\mathbb{R}^m)$ can be generated by (5) with $\delta(t)\equiv 0$.

Proof: See *Appendix B*.

III. Controller Design and Stability Analysis

A. Controller Design

According to (5), to compensate for the disturbance $v(t)$, the FRC is designed as

$$A_{0,\epsilon}\dot{\hat{v}}(t) = -\hat{v}(t) + A_{1,\epsilon}\hat{v}(t-T) + k_e R^T(t)e(t) \quad (6)$$

with $\hat{v}(\theta) = 0$, for $\theta \in [-T, 0]$, where $t \in \mathbb{R}^+$, $A_{0,\epsilon}, A_{1,\epsilon} \in \mathbb{R}^{m\times m}$, $k_e \in \mathbb{R}^+$, and $R(t)$ is as in (3). The FRC (6) can be considered not only as a repetitive controller, but also as a "modified" repetitive controller as described in [1].

1) In particular, the controller (6) with $A_{0,\epsilon} = 0$ and $A_{1,\epsilon} = I_m$ reduces to

$$\hat{v}(t) = \hat{v}(t-T) + k_e R^T(t)e(t). \quad (7)$$

Therefore, the controller (6) with $A_{0,\epsilon} = 0$ is a repetitive controller.

2) Denote the Laplace transforms of $\hat{v}(t)$ and $k_e R^T(t)e(t)$, by $\hat{v}(s)$ and $r_e(s)$ respectively. Then the Laplace transform from $r_e(s)$ to $\hat{v}(s)$ is defined by

$$\hat{v}(s) = \left[1 - Q(s)e^{-sT}\right]^{-1} B(s)r_e(s). \quad (8)$$

Here $Q(s) = (sA_{0,\epsilon} + I)^{-1} A_{1,\epsilon}$ is called the Q-filter in [1] and $B(s) = (sA_{0,\epsilon} + I)^{-1}$. This controller is therefore a "modified repetitive controller" in the sense of [1].

Subtracting (6) from (5) yields

$$A_{0,\epsilon}\dot{\tilde{v}}(t) = -\tilde{v}(t) + A_{1,\epsilon}\tilde{v}(t-T) - k_e R^T(t)e(t) + \delta(t) \quad (9)$$

where $\tilde{v} \triangleq x_p - \hat{v}$. In (9), the initial condition on $\tilde{v}$ is bounded. We do not concern ourselves with the concrete value of the initial condition as the following results hold globally. Combining (1) and (9) yields a closed-loop error dynamics in the form

$$E\dot{z}(t) = f_a(t, z(t)) + f_d(z(t-T)) + b_a(t, z(t))\sigma(t). \quad (10)$$

Here

$$\begin{aligned}
z &= \begin{bmatrix} \tilde{v} \\ e \end{bmatrix} \in \mathbb{R}^{m+n}, \quad \sigma = \begin{bmatrix} \delta \\ 0_{n\times 1} \end{bmatrix} \in \mathbb{R}^{m+n},\\
f_a(t, z(t)) &= \begin{bmatrix} -\tilde{v}(t) - k_e R^T(t)e(t) \\ f(t, e(t)) + b(t, e(t))\tilde{v}(t) \end{bmatrix} \in \mathbb{R}^{m+n},\\
f_d(z(t-T)) &= \begin{bmatrix} A_{1,\epsilon}\tilde{v}(t-T) \\ 0_{n\times 1} \end{bmatrix} \in \mathbb{R}^{m+n},\\
E &= \begin{bmatrix} A_{0,\epsilon} & 0_{m\times n} \\ 0_{n\times m} & I_n \end{bmatrix} \in \mathbb{R}^{(m+n)\times(m+n)},\\
b_a(t, z(t)) &= \begin{bmatrix} I_m & 0_{m\times m} \\ 0_{n\times m} & b(t, e(t)) \end{bmatrix} \in \mathbb{R}^{(m+n)\times 2m}.
\end{aligned}$$

Remark 1: The closed-loop error dynamics (10) is a retarded type system if $A_{0,\epsilon}$ is nonsingular; whereas it will become a neutral type system if $A_{0,\epsilon} = 0$.

B. Stability Analysis

The closed-loop error dynamics (10) will be analyzed with the help of a Lyapunov-Krasovskii functional, with the results stated in *Theorem 1* below. The following preliminary result is needed.

Definition 1 [10]: A solution $z(t, z_{t_0})$ of (10) with $z_{t_0} \in \mathcal{C}([-T, 0]; \mathbb{R}^n)$ is said to be ultimately bounded with bound ϵ, uniformly with respect to $t \geq 0$, if for each $\delta > 0$ there exists $T' = T'(\epsilon, \delta) > 0$ independent of $t_0 \geq 0$ such that, when $\|z_{t_0}\|_c < \delta$, then $\|z(t, z_{t_0})\| \leq \epsilon$ for all $t \geq t_0 + T'$.

Lemma 2 [10]: Assume that there exists a continuously differentiable functional $V(t, z_t) : [0, \infty) \times \mathcal{C}([-T, 0]; \mathbb{R}^n) \to [0, \infty)$ such that

$$\gamma_1 \|z(t)\|^2 \leq V(t, z_t) \leq \gamma_2 \|z(t)\|^2 + \int_{t-T}^{t} \|z(s)\|^2 ds \quad (11)$$

where γ_1, γ_2 are positive real numbers. If there exists $b \in \mathbb{R}^+$ such that

$$\dot{V}(t, z_t) \leq -\gamma_3 \|z(t)\|^2 + b \quad (12)$$

where γ_3 is a positive real number, then the solutions of (10) are uniformly ultimately bounded with respect to the bound $\sqrt{(1+\mu)(b/\gamma_1\gamma_3)(\gamma_2 + T)}$, where μ is an arbitrarily small positive real number.

With the help of *Lemma 2,* we have the following.

Theorem 1: For the error dynamics (1), suppose *Assumptions 1–2* hold, and $\hat{v}(t)$ is generated by the FRC (6) in which the parameters satisfy $k_e > 0$, $A_{0,\epsilon} = A_{0,\epsilon}^T \geq 0$ and $A_{1,\epsilon} = I - \alpha A_{0,\epsilon}$. We claim that 1) if $A_{0,\epsilon} = 0$, then $\lim_{t\to\infty} e(t) = 0$; 2) if $A_{0,\epsilon} > 0$, α ensures $\|A_{1,\epsilon}\| < 1$, and if $V_0(t,e)$ in (2) further satisfies $V_0(t,e) \leq c_1 \|e(t)\|^2$ with $c_1 > 0$, then, for any value of $e(0)$, $e(t)$ is uniformly ultimately bounded.

Proof: For (10), choose the nonnegative function $V(t, z_t)$ to be

$$V(t, z_t) = 2k_e V_0(t, e(t)) + \tilde{v}^T(t)A_{0,\epsilon}\tilde{v}(t) + \int_{t-T}^{t} \tilde{v}^T(s)\tilde{v}(s)ds \quad (13)$$

where V_0 satisfies *Assumption 1* and $A_{0,\epsilon} = A_{0,\epsilon}^T \geq 0$. Under *Assumption 1* taking the time derivative $V(t, z_t)$ along (10) yields

$$\begin{aligned}\dot{V}(t, z_t) \leq &-2k_e e^T(t)M(t)e(t)\\ &+\tilde{v}^T(t)H\tilde{v}(t) + g(t) + 2\tilde{v}^T(t)\delta(t) \quad (14)\end{aligned}$$

where

$$\begin{aligned}
H &= -I_m + A_{1,\epsilon}^T A_{1,\epsilon}\\
g(t) &= -\tilde{v}^T(t)A_{1,\epsilon}^T A_{1,\epsilon}\tilde{v}(t) + 2\tilde{v}^T(t)A_{1,\epsilon}^T\tilde{v}(t-T)\\
&\quad - \tilde{v}^T(t-T)\tilde{v}(t-T).
\end{aligned}$$

Under *Assumptions 1–2,* using the fact that $g(t) \leq 0$, we obtain

$$\dot{V}(t, z_t) \leq -2k_e e^T(t)M(t)e(t) + \tilde{v}^T(t)H\tilde{v}(t) + 2\|\tilde{v}(t)\|\|\delta(t)\|. \quad (15)$$

Based on the results above, conclusions 1) and 2) will be proved in detail in the *Appendix C*. □

IV. Robotic Manipulator Tracking

To show its effectiveness, we apply the proposed controller design to the tracking problem for a class of robotic manipulators. The latter are described by

$$D(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + G(q(t)) + \tau_d(t) = \tau(t) \quad (16)$$

where $q(t) \in \mathbb{R}^n$ is the vector of generalized coordinates, $\tau(t) \in \mathbb{R}^n$ is the vector of input torques, $\tau_d \in \mathcal{C}_{PT}^0([0, \infty); \mathbb{R}^n)$ is the vector of T-periodic disturbances, $D(q(t)) \in \mathbb{R}^{n\times n}$ is the symmetric inertia matrix satisfying $0 < \underline{\lambda}_D I_n \leq D(q(t)) \leq \bar{\lambda}_D I_n < \infty$ for $t \in \mathbb{R}^+$, $C(q(t), \dot{q}(t)) \in \mathbb{R}^{n\times n}$ is the centripetal-Coriolis matrix, and $G(q(t)) \in \mathbb{R}^n$ is the gravity vector.

A. Error Dynamics

In this section, the tracking problem will first be converted into error dynamics as in (1). Define the filtered tracking error as in [11]:

$$e(t) = \dot{\tilde{q}}(t) + \tilde{q}(t)$$

where $\tilde{q}(t) = q_d(t) - q(t)$ and $q_d(t)$ is a desired trajectory. Our control design is based on the fact that the parameters appear linearly in the robot dynamics (16). For convenience of control synthesis, the linear-in-the-parameters property [11] is written as [12]–[14]

$$D(q(t))\ddot{q}_e(t) + C(q(t), \dot{q}(t))\dot{q}_e(t) + G(q(t)) = \Psi(q, \dot{q}, \dot{q}_e, \ddot{q}_e, t)p \tag{17}$$

where $\dot{q}_e(t) = \dot{q}_d(t) + \tilde{q}(t)$ and $\ddot{q}_e(t) = \ddot{q}_d(t) + \dot{\tilde{q}}(t)$, $p \in \mathbb{R}^m$ is the vector of constant parameters, and $\Psi(q, \dot{q}, \dot{q}_e, \ddot{q}_e, t) \in \mathbb{R}^{n\times m}$, denoted by $\Psi(t)$ for brevity. Design $\tau(t)$ according to

$$\tau(t) = M(t)e(t) + \Psi(t)\hat{p}(t) + \hat{\tau}_d(t). \tag{18}$$

Here $M(t) \in \mathbb{R}^{n\times n}$ is a positive definite matrix, $\hat{p}(t)$ is the designer's estimate of p, and $\hat{\tau}_d(t) \in \mathbb{R}^n$ is the designer's estimate of $\tau_d(t)$. By employing (17) and (18), the filtered error dynamics are obtained as follows:

$$\begin{aligned} &D(q(t))\dot{e}(t) + C(q(t), \dot{q}(t))e(t) \\ &\qquad = -M(t)e(t) + \Psi(t)[p - \hat{p}(t)] + [\tau_d(t) - \hat{\tau}_d(t)]. \end{aligned}$$

Furthermore, the system above can be written in the form of (1) with

$$\begin{aligned} f(t, e(t)) &= -D^{-1}(q(t))[C(q(t), \dot{q}(t)) + M(t)e(t)] \\ b(t, e(t)) &= D^{-1}(q(t))[\,\Psi(t) \quad I_n\,] \\ v(t) &= \begin{bmatrix} p \\ \tau_d(t) \end{bmatrix}, \quad \hat{v}(t) = \begin{bmatrix} \hat{p}(t) \\ \hat{\tau}_d(t) \end{bmatrix}. \end{aligned} \tag{19}$$

B. Verification of Assumptions

Define the positive definite function

$$V_0(t, e(t)) = \frac{1}{2}e^T(t)D(q(t))e(t).$$

Then

$$c_0\|e(t)\|^2 \le V_0(t, e(t)) \le c_1\|e(t)\|^2$$

where $c_0 = (1/2)\underline{\lambda}_D$ and $c_1 = (1/2)\bar{\lambda}_D$ are positive reals. By skew-symmetry of matrix $\dot{D}(q(t)) - 2C(q(t), \dot{q}(t))$, the time derivative of $V_0(t, e)$ along (1) with (19) is evaluated as (3) with $R(t) = [\,\Psi(t) \quad I_n\,]$. Therefore, *Assumption 1 is satisfied.* Since p is a vector of constant parameters and $\tau_d(t)$ is the vector of T-periodic disturbances, $v(t)$ in (19) satisfies $v \in \mathcal{C}^0_{PT}([0, \infty); \mathbb{R}^{n+m})$. Therefore, *Assumption 2 is satisfied.*

C. Numerical Simulation

The robot, the initial condition and tracking task used for this example are the same as in [11]. We assume the disturbance is

$$\tau_d(t) = \begin{bmatrix} 0.5\left(1 - \cos\left(\frac{2\pi t}{3}\right)\right) \\ 0.5\sin\left(\frac{2\pi t}{3}t\right) \\ 0.5\left(\cos\left(\frac{2\pi t}{3}\right) - 1\right) \end{bmatrix}$$

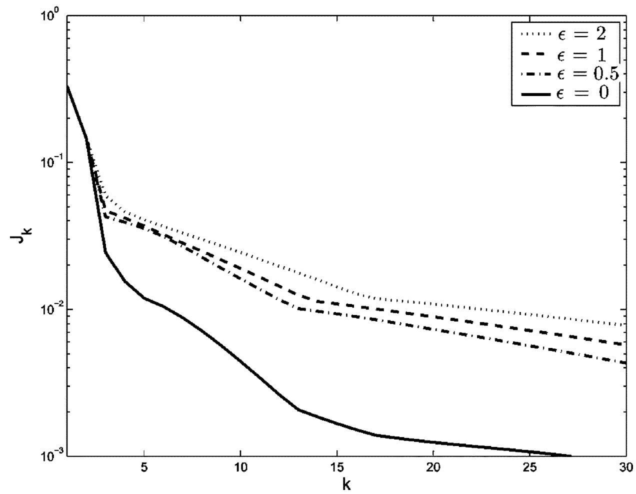

Fig. 2. Tracking performance with different ϵ.

which is unknown for the controller design except for the fact that $\tau_d \in \mathcal{C}^0_{PT}([0, \infty); \mathbb{R}^3)$. According to the dynamics in [11], we obtain the expressions

$$\begin{aligned} p &= [\,J_p \quad l_2 \quad l_3\,]^T \\ \Psi &= \begin{bmatrix} \ddot{q}_{e,1} & 0 & q_3^2\ddot{q}_{e,1} + q_3\dot{q}_3\dot{q}_{e,1} + q_3\dot{q}_1\dot{q}_{e,3} \\ 0 & \ddot{q}_{e,2} + g & \ddot{q}_{e,2} + g \\ 0 & 0 & \ddot{q}_{e,3} - q_3\dot{q}_1\dot{q}_{e,1} \end{bmatrix}. \end{aligned}$$

Here $J_p = 0.8\ \mathrm{kg/m^2}$, $l_2 = 2\ \mathrm{m}$, $l_3 = 1\ \mathrm{m}$, $g = 9.8\ \mathrm{m^2/s}$ and x_j denote the jth element in the vector x, $x = \{q, \dot{q}, \dot{q}_e, \ddot{q}_e\}$. The parameters J_p, l_2, l_3 are assumed unknown for the controller design.

In (18), choose $M(t)$ to be $M(t) \equiv 10I_3$ and design $\hat{v}(t)$ as (6) with $k_e = 1$, $A_{0,\epsilon} = \epsilon I_3$ and $A_{1,\epsilon} = (1 - 0.0001\epsilon)I_3$. For tracking performance comparison, we introduce the performance index as $J_k = \sup_{t\in[(k-1)T, kT]}\|\tilde{q}(t)\|$, where $k = 1, 2, \ldots$.

1) Without Input Delay: If $\epsilon = 0$, then these parameters above satisfy the conditions of *Theorem 1*. As seen in Fig. 2, the performance index J_k approaches 0 as k increases. It implies that $\tilde{q}(t)$ approaches 0 as $t \to \infty$. This result is consistent with conclusion 1) in *Theorem 1*. If $\epsilon = 2$, 1, 0.5, then the chosen parameters satisfy the conditions of *Theorem 1*. As seen in Fig. 2, the performance index J_k is bounded. This result is consistent with conclusion 2) in *Theorem 1*.

As seen in Fig. 2, the tracking performance improves as ϵ decreases. This is consistent with the conclusion for linear systems that, recalling (8), as the bandwidth of $Q(s) = ((1 - 2\epsilon)/(\epsilon s + 1))I_3$ increases, i.e., ϵ decreases, the tracking performance improves, and vice versa [1], [4].

2) With Input Delay: Consider the robotic manipulator with an input delay as follows:

$$D(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t))\dot{q}(t) + G(q(t)) + \tau_d(t) = \tau(t - d_\varepsilon)$$

where $d_\varepsilon > 0$ is the input delay. The controllers are designed as above. When RC is adopted ($\epsilon = 0$) and $d_\varepsilon = 0.001$ s, it is observed from Fig. 3 that a state of the closed-loop system diverges at about 30 s. This confirms that the stability of RC systems is insufficiently robust. Using the FRC with $\epsilon = 0.5$ and prolonging the input delay from $d_\varepsilon = 0.001$ s to $d_\varepsilon = 0.01$ s, it is observed from Fig. 3 that the closed-loop system can still track the desired trajectory. This shows that an FRC system is more robust than its corresponding RC system.

Remark 2: From the simulations, the FRC provides the flexibility to choose filter parameters to achieve a tradeoff between tracking performance and stability. More importantly, it is shown that FRC is effective because input delay exists widely in practice.

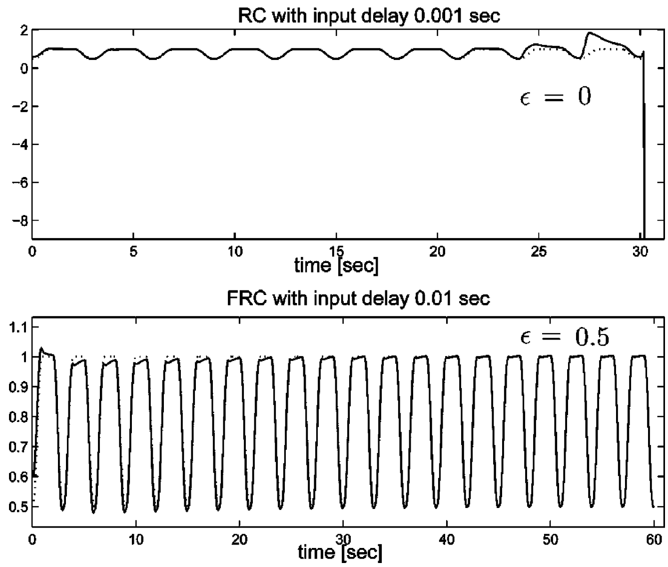

Fig. 3. Time response of $q_{d,3}(t)$ and $q_3(t)$ with input delay. [solid-response, dotted-desired trajectory].

V. Conclusion

Compared with existing repetitive controllers, the FRC provides flexibility to choose different filter parameters such as $A_{0,\epsilon}$ to satisfy performance requirements and tolerate some small uncertainties, such as input delay. If a periodic disturbance is present, then the proposed controller with $A_{0,\epsilon} = 0$ causes the tracking error to approach 0 as $t \to \infty$. In order to demonstrate its effectiveness, the proposed method is applied to periodic disturbance rejection in a class of robotic manipulators. Simulation data show that a tradeoff between tracking performance and stability can be achieved by tuning the filter parameters. Furthermore, simulation data show that FRC can deal with small input delay while the corresponding RC cannot, which demonstrates the effectiveness of FRC.

Appendix A
RC Systems With an Input Delay

Consider the following simple RC system:

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t) \\ u(t) &= u(t-T) - x(t) \end{aligned} \tag{20}$$

where $x(t), u(t) \in \mathbb{R}$. The RC system above can be also written as

$$\dot{x}(t) - x(t-T) = -2x(t) + x(t-T). \tag{21}$$

From [6], the system above is asymptotically stable. In the following, we claim that the RC system (20) with an input delay $\tau_\varepsilon > 0$, namely

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t-\tau_\varepsilon) \\ u(t) &= u(t-T) - x(t) \end{aligned} \tag{22}$$

will lose its stability no matter how small τ_ε is.

For simplicity, let $\tau_\varepsilon = T/2N$, where N is a positive integer. The characteristic equation of (22) is

$$1 - e^{-Ts} = -e^{-(T/2N)s}\frac{1}{s+1}. \tag{23}$$

From [15, p. 28, Lemma 7.1], the roots of (23) will approach $\pm(2k\pi/T)j$ as $k \to \infty$. Therefore, if k is sufficiently large, then a root of (23) is

$$s^* = \frac{2k\pi}{T}j + \alpha_\varepsilon + \beta_\varepsilon j$$

with sufficiently small $\alpha_\varepsilon, \beta_\varepsilon \in \mathbb{R}$. Setting $k = 2Nk' + N$, where k' is a positive integer, and substituting s^* into (23) results in

$$\begin{aligned} &1 - e^{-T\alpha_\varepsilon}\left(\cos(T\beta_\varepsilon) - j\sin(T\beta_\varepsilon)\right) \\ &\qquad = e^{-(T/2N)\alpha_\varepsilon}\frac{\cos\left(\frac{T}{2N}\beta_\varepsilon\right) - j\sin\left(\frac{T}{2N}\beta_\varepsilon\right)}{\frac{2k\pi}{T}j + \alpha_\varepsilon + \beta_\varepsilon j + 1}. \end{aligned}$$

Setting k' to be large enough, and then neglecting the higher order infinitesimal, we have

$$\begin{aligned} 1 - e^{-T\alpha_\varepsilon}\cos(T\beta_\varepsilon) &\approx e^{-(T/2N)\alpha_\varepsilon}\frac{1 - \sin\left(\frac{T}{2N}\beta_\varepsilon\right)\frac{2k\pi}{T}}{\left(\frac{2k\pi}{T}\right)^2} \\ e^{-T\alpha_\varepsilon}\sin(T\beta_\varepsilon) &\approx -\frac{e^{-(T/2N)\alpha_\varepsilon}}{\frac{2k\pi}{T}}. \end{aligned}$$

Furthermore, we can obtain

$$\alpha_\varepsilon \approx \frac{T(N+1)}{2N(2k\pi)^2}.$$

Therefore, we can conclude $\alpha_\varepsilon > 0$. This implies the RC system (20) with an input delay $\tau_\varepsilon > 0$ is unstable no matter how small τ_ε is (or how large N is).

Appendix B
Proof of Lemma 1

Let (4) be the original system. To apply *Additive Decomposition* [16], choose the primary system as follows:

$$\begin{aligned} A_{0,\epsilon}\dot{x}_p(t) &= -x_p(t) + A_{1,\epsilon}x_p(t-T) + \delta(t) \\ x_p(\theta) &= v(T+\theta), \theta \in [-T, 0] \end{aligned} \tag{24}$$

where $A_{0,\epsilon}, A_{1,\epsilon} \in \mathbb{R}^{n\times n}$, $\delta(t) \in \mathbb{R}^n$. From the original system (4) and the primary system (24), the secondary system is determined by *Additive Decomposition* [16]:

$$\begin{aligned} -A_{0,\epsilon}\dot{x}_p(t) &= -x_s(t) + x_s(t-T) + (I - A_{1,\epsilon})x_p(t-T) - \delta(t) \\ x_s(\theta) &= 0, \qquad \theta \in [-T, 0]. \end{aligned} \tag{25}$$

By *Additive Decomposition* [16], we have

$$x(t) = x_p(t) + x_s(t). \tag{26}$$

Let $\delta(t) \equiv x_s(t-T)$; then (4) can be written as

$$\begin{aligned} A_{0,\epsilon}\dot{x}_p(t) &= -x_p(t) + A_{1,\epsilon}x_p(t-T) + \delta(t) \\ v(t) &= x_p(t) + \delta(t+T) \\ x_p(\theta) &= v(T+\theta), \qquad \theta \in [-T, 0]. \end{aligned} \tag{27}$$

In the following, we will discuss the bound on $\delta(t), t \in [0, \infty)$. From (25) and (26), we obtain

$$A_{0,\epsilon}\dot{x}_p(t) = -x_p(t) - (I - A_{1,\epsilon})x_p(t-T) + v(t). \tag{28}$$

If $A_{0,\epsilon} > 0$ and $\|A_{1,\epsilon}\| < 1$, then $A_{0,\epsilon}\dot{x}_p(t) = -x_p(t) - (I - A_{1,\epsilon})x_p(t-T)$ is exponentially stable[17]. Consequently, there exists a bounded and T-periodic function $x_p(t)$ which satisfies

(28)[18]. Note that $\delta(t) \equiv v(t-T) - x_p(t-T)$ is also a bounded and T-periodic function. Consequently, (27) can be rewritten in the form of (5).

If $A_{0,\epsilon} = 0$ and $A_{1,\epsilon} = I_m$, then (4) can be rewritten as (5) with $\delta(t) \equiv 0$.

Appendix C
Detailed Proof of Theorem 1

1) If $A_{0,\epsilon} = 0$, then $\delta(t) \equiv 0$ by *Lemma 1*. Thus (15) becomes

$$\dot{V}(t, z_t) \leq -2k_e e^T(t) M(t) e(t). \tag{29}$$

Then

$$\dot{V}(t, z_t) \leq -2c_3 \|e(t)\|^2.$$

From the inequality above, we obtain

$$V(t, z_t) \leq V(0, z_0). \tag{30}$$

Also

$$\int_0^t \|e(s)\|^2 ds \leq \frac{1}{2c_3} V(0, z_0).$$

Therefore, $e \in \mathcal{L}_\infty \cap \mathcal{L}_2$. Next, we prove that $e(t)$ is uniformly continuous. If this is true, then $\lim_{t\to\infty} e(t) = 0$ by Barbalat's Lemma [19].

Since $e \in \mathcal{L}_\infty$, $e(t) \in K \subset \mathbb{R}^m$ for a compact set K. By *Assumption 2,* there exists $b_f \in \mathbb{R}^+$ such that $\|f(t,e)\| \leq b_f$ for all $(t, e(t)) \in [0,\infty) \times K$. Let t_1 and t_2 be any real numbers such that $0 < t_2 - t_1 \leq h$. Then we have

$$\begin{aligned} \|e(t_2) - e(t_1)\| &= \left\| \int_{t_1}^{t_2} \dot{e}(s) ds \right\| \\ &= \left\| \int_{t_1}^{t_2} [f(s, e(s)) + b(s, e(s)) \tilde{v}(s)] ds \right\| \\ &\leq b_f (t_2 - t_1) + b_b \int_{t_1}^{t_2} \|\tilde{v}(s)\| ds. \end{aligned} \tag{31}$$

Note that by (30) $V(t, z_t)$ is bounded for all $t \in \mathbb{R}^+$ with respect to the bound $b_V \in \mathbb{R}^+$, hence

$$\int_{t-T}^{t} \|\tilde{v}(s)\|^2 ds \leq b_V$$

for all $t \in \mathbb{R}^+$. Thus

$$\int_{t_1}^{t_2} \|\tilde{v}(s)\|^2 ds \leq N b_V \tag{32}$$

where $N = \lfloor (t_2 - t_1)/T \rfloor + 1$ and $\lfloor (t_2 - t_1)/T \rfloor$ represents the floor integer of $(t_2 - t_1)/T$. By the Cauchy–Schwarz inequality, we have

$$\int_{t_1}^{t_2} \|\tilde{v}(s)\| ds \leq \left(\int_{t_1}^{t_2} 1^2 ds \right)^{1/2} \left[\int_{t_1}^{t_2} \|\tilde{v}(s)\|^2 ds \right]^{1/2}.$$

Consequently, $\|e(t_2) - e(t_1)\|$ in (31) is further bounded by

$$\|e(t_2) - e(t_1)\| \leq b_f h + h_1 \sqrt{h}$$

where (32) is utilized and $h_1 = \sqrt{N b_V} b_b$. Therefore, for any $\varepsilon > 0$ there exists

$$h = \left[\frac{-h_1 + \sqrt{h_1^2 + b_f \varepsilon}}{2b_f} \right]^2 > 0$$

such that $\|e(t_2) - e(t_1)\| < \varepsilon$ for any $0 < t_2 - t_1 < h$. This implies that $e(t)$ is uniformly continuous.

2) By noticing the form of $V(t, z_t)$ in (13), the inequality (11) is satisfied with $\gamma_1 = \min[2k_e c_1, \lambda_{\min}(A_{0,\epsilon})]$ and $\gamma_2 = \max[2k_e c_2, \lambda_{\max}(A_{0,\epsilon})]$. If $A_{0,\epsilon} > 0$, then $\delta \in \mathcal{L}_\infty$ by *Lemma 2*. Then, the inequality (15) becomes

$$\dot{V}(t, z_t) \leq -2\kappa_1 \|z(t)\|^2 + 2\kappa_2 \|z(t)\|$$

where $\kappa_1 = \min[\lambda_M k_e, (1/2)\lambda_{\min}(A_{0,\epsilon})]$ and $\kappa_2 = \|\delta\|_\infty$. Since $\kappa_1 > 0$, the following inequality holds:

$$-\kappa_1 \|z(t)\|^2 + 2\kappa_2 \|z(t)\| - \kappa_1^{-1} \kappa_2^2 \leq 0.$$

Therefore (12) is satisfied with $\gamma_3 = \kappa_1$ and $b = \kappa_1^{-1}\kappa_2^2$. Note the fact that $\|e(t)\| \leq \|z(t)\|$ by the definition of $z(t)$, then $e(t)$ is globally uniformly ultimately bounded with the ultimate bound

$$\sqrt{\frac{(1+\mu)(\rho+1)^2 \|\delta\|_\infty^2 \{\max[2k_e c_2, \lambda_{\max}(A_{0,\epsilon})] + T\}}{\min[2k_e c_1, \lambda_{\min}(A_{0,\epsilon})] \left(\min\left[\lambda_M k_e, \frac{1}{2}\lambda_{\min}(A_{0,\epsilon})\right]\right)^2}} \tag{33}$$

where μ is an arbitrarily small positive real number.

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A Distributed Algorithm for Proportional Task Allocation in Networks of Mobile Agents

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***Abstract*—In a proportional task allocation problem, it is desired for robotic agents to have equal duty to capability ratios. Here, this problem is addressed as a combination of deployment and consensus problems. Tasks occur in a convex region and each task is assigned to its nearest agent. Agents are deployed on this area in order to reach consensus over the value of their duty to capability ratio. A distributed, asynchronous, and scalable algorithm is presented for solving this problem in continuous time domain.**

***Index Terms*—Consensus, deployment, proportional task allocation, Voronoi partition.**

I. INTRODUCTION

In a *proportional task allocation problem* (also known as *equitable location problem* in operations research literature [1], [2]), it is desired for the agents to have equal duty to capability ratios. In other words, the agents with more capability must perform more tasks, and vice versa. A few applications of this problem include balancing load on agents (which slows down their wearing and maximizes their lifetime), minimizing the waiting time for receiving service at a facility, and sharing computational resources in real-time systems (e.g., multimedia systems).

In this technical note, an agent's capability is considered as a measure of the number of tasks that it can execute simultaneously. Also, it is assumed that an agent regardless of its capability can execute any task. Thus, the phrase "agent a is less capable than agent b" means that in equal time spans, agent b can perform more tasks than agent a (e.g. due to possession of more resources). The following example demonstrates a proportional task allocation problem.

Example 1: Consider each agent as a *movable base station (MBS)* [3] in a wireless communication network. Here, the capability of an agent is measured by the number of simultaneous calls that it can handle. The agent's duty is represented by the calls that are made in the dominance region of that agent (i.e. the MBS's cell). In this example, the goal is partitioning a region and assigning each part to a station in a way that the duty to capability ratio is equal for all communication stations. It is assumed that the range of each MBS is enough for a good coverage over its assigned cell. This is the case in densely populated areas, where the constraint is imposed on the possibility of making simultaneous calls rather than the coverage range of the antennas. Solving this proportional task allocation problem minimizes the probability of connection blockings and equalizes the load on the communication stations.

To the best of the authors' knowledge, proportional task allocation problem was scarcely studied before. In [1], a centralized solution for a special setting of this problem is introduced: a group of facilities are positioned on the unit square with uniform demand density such that the maximal demand faced by each facility is minimized. The work in [4] proposes a centralized algorithm for dividing a region into equal areas. The algorithm is based on additively weighted power diagrams (a generalization of *Voronoi diagrams*) and uses Newton's method for updating these weights. However, convergence is not guaranteed and depends on initial weights and locations of the agents. Due to the centralized nature of these algorithms [1], [2], [4], their application is limited to the networks with small number of agents and static environmental constraints.

In a recent work [5], a distributed algorithm is presented based on the notion of power diagrams. The proposed local controller solves the task allocation problem for the special case when all agents have equal capabilities. In this algorithm the generators of the power diagrams are fixed and the controller updates the weights of the generators. A general drawback of power diagrams is that the generators can be located out of their dominance region. Moreover, the assumption of fixed generators makes the algorithm inapplicable to deployment problems.

Occasionally, deployment problems can be construed as task assignment problems, and vice versa. For instance, the goal in the deployment problem of [6] is districting a convex demand region by assigning each demand point to an agent (considered as a facility), so that the summation of squared distances between any demand point and its assigned facility is minimal. Based on this speculation, the formulation available in locational optimization and deployment literature is modified to solve the proportional task allocation problem.

In this technical note, the proportional task allocation problem is regarded as a deployment problem in which the optimal coverage is achieved only when the agents reach consensus on the value of their duty to capability ratio. The main contribution of this work is combining the notions of consensus and deployment to propose a distributed, asynchronous, and scalable algorithm for solving the proportional task allocation problem in multi-agent systems.

The outline of this technical note is as follows. Section II reviews a few notions related to Voronoi partitions. The proportional task allocation problem is mathematically modeled using an aggregate objective function in Section III. Smoothness properties of the aggregate objective function are analyzed in Section IV. In Section V, a continuous-time distributed algorithm is presented that asymptotically solves the proportional task allocation problem. Scalability and asynchronism of the algorithm are also discussed in this section. Simulation results are presented in Section VI. Section VII states the concluding remarks.

II. VORONOI PARTITIONS AND DELAUNAY GRAPHS

The following definitions and notions on Voronoi partitions and Delaunay graphs are taken from [6], [7].

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