INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust. Nonlinear Control 2012; **22**:1873–1880
Published online 21 September 2011 in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/rnc.1793

SHORT COMMUNICATION

A new method to obtain ultimate bounds and convergence rates for perturbed time-delay systems

Quan Quan*,†‡ and Kai-Yuan Cai

Department of Automatic Control, Beihang University (Beijing University of Aeronautics and Astronautics), Beijing 100191, China

SUMMARY

A new method is proposed to determine the ultimate bounds and the convergence rates for perturbed time-delay systems when the Lyapunov–Krasovskii functionals and their derivatives are available. Compared with existing methods, the proposed method is more concise, more widely applicable, and the obtained results are less conservative. To show the three features, the proposed method is applied to improve three existing results, respectively. Copyright © 2011 John Wiley & Sons, Ltd.

Received 1 January 2011; Revised 17 May 2011; Accepted 4 July 2011

KEY WORDS: boundedness; time-delay systems; Lyapunov–Krasovskii functionals

1. INTRODUCTION

The problem of determining system stability in the presence of nonvanishing perturbations, that is, perturbations that do not vanish as the state approaches an equilibrium point, has attracted much attention [1–8]. The nonvanishing perturbations could result from modeling errors, uncertainties and disturbances, and others. It is pointed out in [2] that in the presence of such perturbations, asymptotic stability is in general not possible, but the ultimate boundedness of the system's trajectories can be achieved. In this case, it is important to determine ultimate bounds and convergence rates of the systems' trajectories as a measure of the controller performance. For perturbed time-delay systems, the existing methods are often based on the use of quadratic Lyapunov functions to determine ultimate bounds and convergence rates [4–7]. It is well known that simple quadratic Lyapunov functions often lead to the conservative ultimate boundedness criteria for time-delay systems. As a result, Xu and Feng in [8] designed a Lyapunov–Krasovskii functional to replace the quadratic Lyapunov function proposed in [6] for the same problem. It is shown that the ultimate boundedness criterion in [6] is improved in [8]. However, by following the method proposed in [8], other inequality techniques are used after obtaining the Lyapunov–Krasovskii functional and its derivative. This still leads the obtained ultimate bound to conservatism. This is our first motivation.

The problem of determining system stability in the presence of nonvanishing perturbations can be also formulated in the paradigm of input to state stability (ISS) [9], where nonvanishing perturbations are considered as the input. For time-delay systems, Teel proposed sufficient conditions in the setting of Lyapunov–Razumikhin theorems to yield the ISS [10]. Furthermore, Pepe and Jiang

*Correspondence to: Quan Quan, Department of Automatic Control, Beihang University (Beijing University of Aeronautics and Astronautics), Beijing 100191, China.

†E-mail: qq_buaa@asee.buaa.edu.cn

‡Quan Quan is also with the State Key Laboratory of Virtual Reality Technology and Systems, Beihang University, Beijing 100191, China

addressed ISS from a perspective of Lyapunov–Krasovskii functionals for time-delay systems [11]. However, the proposed sufficient conditions cannot be applied to the case here. This is our second motivation.

In this paper, we propose a new method to determine the ultimate bounds and the convergence rates for perturbed time-delay systems on the basis of Lyapunov–Krasovskii functionals and their derivatives. Compared with existing methods, the proposed method has the following salient features.

- *More concise*. The proof of a theorem, which is used to study the uniform ultimate boundedness in [3], is simplified by the proposed method. In addition, the convergence rate of the solutions is given.
- *More widely applicable*. The proposed method can handle complicated Lyapunov–Krasovskii functionals. As an application, on the basis of the results obtained in [5], it is proven that the solutions are uniformly ultimately bounded instead of the uniform boundedness as asserted in [5].
- *Less conservative*. Compared with the ultimate bound obtained in [8], it is proven that a tighter ultimate bound is derived by the proposed method.

Besides these salient features, the proposed method is given through a theorem that can be applied to other applications easily. The contributions of this paper are as follows: (i) a new method with salient features to determine the ultimate bounds and the convergence rates for perturbed time-delay systems and (ii) the ability of the proposed method to cope with some cases whereas existing methods for ISS cannot.

Notation used in this paper is as follows. The symbol $\mathbb{R}^n$ is the Euclidean space of dimension n. Let $\mathbb{R}^+$ denote the positive real numbers and $\mathbb{N}$ denote the positive integral numbers. The symbol $\|\cdot\|$ stands for the Euclidean norm or the induced Euclidean norm. The symbol $C_{n,\tau}$ denotes the space of continuous n-dimensional vector functions on $[-\tau, 0]$. $x_t \triangleq x(t+s), s \in [-\tau, 0], \tau \in \mathbb{R}^+$. The symbol $\lambda_{\min}(X)$ denotes the minimum eigenvalue of matrix X.

2. PROBLEM FORMULATION

Consider a general perturbed time-delay system

$$\dot{x}(t) = F(t, x_t, w) \tag{1}$$

with $x(s) = \phi(s), s \in [-\tau, 0], \tau \in \mathbb{R}^+$, where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$ is a Lebesgue measurable and bounded perturbation. $F : \mathbb{R}^+ \cup \{0\} \times C_{n,\tau} \times \mathbb{R}^m \to \mathbb{R}^n$ is supposed to ensure the existence and uniqueness of the solution through every initial condition ϕ.

Definiton 1 ([3]*)*
The solutions $x_t(\phi)$ of system (1) with $x(t_0+s) = \phi(s), s \in [-\tau, 0]$ are said to be *uniformly ultimately bounded* with *ultimate bound* ε, if for each $\delta > 0$ there exists $T = T(\varepsilon, \delta) > 0$ independent of t_0 such that $\|x(\phi)(t)\| \leqslant \varepsilon$ for all $t \geqslant t_0 + T$ when $\sup_{s\in[-\tau,0]} \|\phi(s)\| < \delta$.

The purpose of this paper is to propose a new method to determine the ultimate bound and the convergence rate for Equation (1) when the Lyapunov–Krasovskii functionals and their derivatives along the solutions of Equation (1) are available.

Before proceeding further with the development of this paper, the following preliminary result is needed.

Lemma 1 ([4]*)*
Let $g(t)$ be a continuous function with $g(t) \geqslant 0$ for all $t \geqslant t_0 - r_0$ and $k_0 > \sup_{s\in[-r_0,0]} g(t_0+s)$. Let $\dot{g}(t) \leqslant -\alpha_1 g(t) + \alpha_2 \sup_{s\in[-r_0,0]} g(t+s) + \beta$ for $t \geqslant t_0$ where $r_0, \alpha_1, \alpha_2, \beta \in \mathbb{R}^+$. If $\alpha_1 > \alpha_2$, then $g(t) \leqslant g_0 + k_0 e^{-\lambda_0(t-t_0)}$, for $t \geqslant t_0$, where $g_0 = \beta/(\alpha_1 - \alpha_2)$ and λ_0 is the unique solution to the equation $-\lambda_0 = -\alpha_1 + \alpha_2 e^{\lambda_0 r_0}$.

3. MAIN RESULTS

In this section, the new method is proposed through a theorem. The theorem can help to determine the ultimate bounds and the convergence rates for perturbed time-delay systems when the Lyapunov–Krasovskii functionals and their derivatives are available. With the help of Lemma 1, we have

Theorem 1
Suppose that there exists a Lyapunov–Krasovskii functional $V(t, x_t) : \mathbb{R}^+ \cup \{0\} \times C_{n,\tau} \to \mathbb{R}^+ \cup \{0\}$ such that

$$
\begin{aligned}
\gamma_1 \|x(t)\|^2 &\leqslant V(t, x_t) \\
&\leqslant \gamma_2 \|x(t)\|^2 + \sum_{i=1}^{N_1} \rho_i \int_{-\tau_i}^{0} \|x_t(s)\|^2 \mathrm{d}s \\
&\quad + \sum_{i=1}^{N_1'} \sum_{j=1}^{N_2'} \rho_{ij}' \int_{-\tau_i'}^{0} \int_{-\tau_j''+s}^{0} \|x_t(\xi)\|^2 \mathrm{d}\xi \mathrm{d}s + c
\end{aligned} \tag{2}
$$

where $N_1, N_1', N_2' \in \mathbb{N}$, $\gamma_1, \gamma_2, c, \rho_i, {\rho'}_{ij}, \tau_i, \tau_i', \tau_j'' \in \mathbb{R}^+ \cup \{0\}$, and there exists a $\sigma \in \mathbb{R}^+ \cup \{0\}$ such that

$$
\dot{V}(t, x_t) \leqslant -\gamma_3 \|x(t)\|^2 + \sigma, \ \gamma_3 \in \mathbb{R}^+ \tag{3}
$$

where $\dot{V}(t, x_t)$ is the derivative of $V(t, x_t)$ along the solutions of Equation (1). Then (i) $V(t, x_t)$ satisfies

$$
V(t, x_t) \leqslant v_0 + k e^{-\lambda t} \tag{4}
$$

where λ is the unique solution to the equation

$$
-\lambda = -\frac{v_1 + v_2}{v_3} - \frac{1}{v_3} + \frac{v_1 + v_2}{v_3} e^{\lambda \tau}, \tag{5}
$$

$k > \sup_{s \in [-\tau, 0]} \|V(s, x_s)\|$ and

$$
\begin{aligned}
r_1 &= \max_i (\tau_i), r_2 = \max_{i,j} \left(\tau_i' + \tau_j''\right), \tau = \max(r_1, r_2) \\
v_0 &= \sigma \frac{\gamma_2}{\gamma_3} + \frac{\sigma}{\gamma_3} \left[\sum_{i=1}^{N_1} \rho_i \tau_i + \sum_{i=1}^{N_1'} \sum_{j=1}^{N_2'} \rho_{ij}' \left(\tau_i' \tau_j'' + \frac{1}{2} \tau_i'^2 \right) \right] + c \\
v_1 &= \sum_{i=1}^{N_1} \frac{\rho_i}{\gamma_3}, v_2 = \sum_{i=1}^{N_1'} \sum_{j=1}^{N_2'} \frac{{\rho'}_{ij}}{\gamma_3} \tau_i', v_3 = \frac{\gamma_2}{\gamma_3}.
\end{aligned} \tag{6}
$$

(ii) $\|x(t)\|$ satisfies

$$
\|x(t)\| \leqslant \sqrt{\frac{v_0}{\gamma_1}} + \sqrt{\frac{k}{\gamma_1}} e^{-\frac{\lambda}{2} t}, \ \gamma_1 \in \mathbb{R}^+, \tag{7}
$$

which implies that the solutions of Equation (1) are uniformly ultimately bounded with ultimate bound $[v_0/\gamma_1]^{1/2} + \epsilon$, where $\epsilon \in \mathbb{R}^+$ is an arbitrarily small number.

Proof
Let $V(t) = V(t, x_t)$ for simplicity. From Equation (3), we can obtain

$$
\|x(t)\|^2 \leqslant \frac{\sigma - \dot{V}(t)}{\gamma_3}. \tag{8}
$$

DOI: 10.1002/rnc

Substituting Equation (8) into Equation (2) yields

$$\begin{aligned} V(t) &\leqslant \frac{\gamma_2}{\gamma_3}\left[\sigma - \dot{V}(t)\right] + \sum_{i=1}^{N_1} \rho_i \int_{-\tau_i}^{0} \frac{\sigma - \dot{V}(t+s)}{\gamma_3} \mathrm{d}s \\ &\quad + \sum_{i=1}^{N_1'} \sum_{j=1}^{N_2'} \rho_{ij}' \int_{-\tau_i'}^{0} \int_{-\tau_j''+s}^{0} \frac{\sigma - \dot{V}(t+\xi)}{\gamma_3} \mathrm{d}\xi \mathrm{d}s + c. \end{aligned} \tag{9}$$

Note that

$$-\sum_{i=1}^{N_1} \frac{\rho_i}{\gamma_3} \int_{-\tau_i}^{0} \dot{V}(t+s)\,\mathrm{d}s \leqslant -v_1 V(t) + v_1 \sup_{s\in[-r_1,0]} V(t+s)$$

and

$$-\sum_{i=1}^{N_1'} \sum_{j=1}^{N_2'} \frac{\rho'_{ij}}{\gamma_3} \int_{-\tau_i'}^{0} \int_{-\tau_j''+s}^{0} \dot{V}(t+\xi)\,\mathrm{d}\xi \mathrm{d}s \leqslant -v_2 V(t) + v_2 \sup_{s\in[-r_2,0]} V(t+s),$$

then the right-hand side of inequality (2) becomes

$$\begin{aligned} V(t) &\leqslant -v_3 \dot{V}(t) - v_1 V(t) + v_1 \sup_{s\in[-r_1,0]} V(t+s) \\ &\quad - v_2 V(t) + v_2 \sup_{s\in[-r_2,0]} V(t+s) + v_0 \end{aligned}$$

where v_0, v_1, v_2, v_3 are defined in equation (6). Rearranging this inequality results in

$$\dot{V}(t) \leqslant -\frac{v_1 + v_2 + 1}{v_3} V(t) + \frac{v_1 + v_2}{v_3} \sup_{s\in[-\tau,0]} V(t+s) + \frac{v_0}{v_3}. \tag{10}$$

Note that $V(t)$ is a continuous function with $V(t) \geqslant 0$ for all $t \geqslant -\tau$. Then by Lemma 1, we have Equation (4), where $k > \sup_{s\in[-\tau,0]} \|V(s)\|^2$ and λ satisfies Equation (5). Furthermore, if $\gamma_1 \|x(t)\|^2 \leqslant V(t)$ with $\gamma_1 \in \mathbb{R}^+$, then inequality (7) is satisfied. From Definition 1, we can conclude this proof. □

Remark 1
The constant c used in Equation (2) is to enlarge the class of functionals. For example, the constant c can represent the upper bound of some variable independent of the state or some bounded functionals such as $\int_{t-\tau}^{t} \mathrm{sat}^T(x(s))\mathrm{sat}(x(s))\,\mathrm{d}s$, where $\mathrm{sat}(\cdot)$ denotes the saturated term.

Remark 2
The involved bounds here are only functionals of the quadratic type, a special case of $\mathcal{K}_\infty$ functions. On the other hand, existing methods for ISS often handle more general Lyapunov functionals whose bounds are represented by $\mathcal{K}_\infty$ functions. However, the former reserves some structured information of Lyapunov functionals, namely the quadratic type, whereas the latter has lost. Just because of the use of the structure, some salient features of the proposed method are derived.

Remark 3
The existing sufficient conditions in the setting of Lyapunov–Razumikhin theorems or Lyapunov–Krasovskii functionals are often in the form [10, 11]:

$$\alpha_1(\|x_t(0)\|) \leqslant V(x_t) \leqslant \alpha_2(\|x_t\|_a) \tag{11}$$

$$\dot{V}(x_t) \leqslant -\alpha_3(\|x_t\|_a) + \sigma, \tag{12}$$

DOI: 10.1002/rnc

where $\|\cdot\|_a$ denotes a norm on the space $C_{n,\tau}$, and $\alpha_1, \alpha_2, \alpha_3$ are $\mathcal{K}_\infty$ functions (or α_3 is a $\mathcal{K}$ function). By eliminating $\|x_t\|_a$ in Equation (12), we can obtain

$$\dot{V}\left(x_t\right) \leqslant -\left(\alpha_3 \circ \alpha_2^{-1}\right)\left(V\left(x_t\right)\right) + \sigma. \tag{13}$$

By employing equation (13), ISS or the property of uniformly ultimate boundedness can be obtained. However, the Lyapunov–Krasovskii functional with the properties (2) and (3) is in the form

$$\begin{aligned} \alpha_1\left(\left\|x_t\left(0\right)\right\|\right) \leqslant V\left(x_t\right) &\leqslant \alpha_2\left(\left\|x_t\right\|_a\right) \\ \dot{V}\left(x_t\right) &\leqslant -\alpha_3\left(\left\|x_t\left(0\right)\right\|\right) + \sigma, \end{aligned} \tag{14}$$

where $\|\cdot\|_a$ differs from $\|\cdot\|$ that the former is defined on the space $C_{n,\tau}$, whereas the latter is defined on the space $\mathbb{R}^n$. There does not exist positive constants c_1 and c_2 such that $c_1\left\|x_t\left(0\right)\right\| \leqslant \left\|x_t\right\|_a \leqslant c_2\left\|x_t\left(0\right)\right\|$. Without this, it is difficult to obtain inequality (13) only from equation (14). Consequently, it is difficult to obtain ISS or the property of uniformly ultimately bounded by using existing methods for ISS.

Remark 4

If the disturbance w or say the input is only measurable and locally essentially bounded as in [11, 12] and (1) holds almost everywhere, then Equation (3) may only hold almost everywhere, where σ depends on $\|w\|$. It is not a sufficient condition for a (simply) continuous function to be nonincreasing if the upper right-hand Dini derivative of such function is nonpositive almost everywhere [12]. The condition will be sufficient if the function is locally absolutely continuous. Taking this into account, Pepe in [12] discussed these problems and gave sufficient conditions to check the absolute continuity of a Lyapunov–Krasovskii functional. If the considered Lyapunov–Krasovskii functional is locally absolutely continuous, then by the similar idea as in [11, 12], the conclusions obtained in this paper hold almost everywhere.

4. APPLICATIONS

To demonstrate the effectiveness, the proposed method is used to improve three existing results. The system (1) is a general perturbed time-delay system that can represent all the closed-loop systems mentioned here. For clarity, some symbols used in the following are different from those in the original literature.

4.1. Application 1 (more concise)

The literature [3] proposed a theorem (Theorem 1 in [3]) to investigate the uniform ultimate boundedness of perturbed time-delay systems. However, its proof, which is based on stability definitions, is somewhat complicated. As a result, it might be difficult to extend this idea of the proof to the case where $V\left(t, x_t\right)$ has a complicated form. Moreover, the convergence rate of the solutions is difficult to obtain. Let $c = 0, \tau_1 = \tau, \rho_1 = 1, N_1 = 1, N_1' = N_2' = 0$ in Equation (2), then the proposed theorem can reduce to Theorem 1 in [3]. In addition, compared with Theorem 1 in [3], the proposed theorem gives the convergence rate of the solutions.

4.2. Application 2 (more widely applicable)

Consider the time-delay system of the form [5]

$$\dot{x}\left(t\right) = Ax\left(t\right) + A_d x\left(t-\tau\right) + Bsat\left(u\left(t\right)\right) + Ef\left(t\right) \tag{15}$$

with

$$x\left(s\right) = \phi\left(s\right), s \in \left[-\tau, 0\right]$$

DOI: 10.1002/rnc

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and $sat : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a saturation function. The matrices A, A_d, B, and E are constant matrices of appropriate dimensions. The vector $f(t)$ is an external disturbance vector bounded by $\|f(t)\| \leqslant \sigma_f$.

The literature [5] proposed a controller to stabilize a class of dynamic systems subject to time delays, external disturbances, and control saturations. For stability analysis, the Lyapunov–Krasovskii functional in [5] is designed as

$$\begin{aligned} V(t, x_t) = x^T(t) P x(t) &+ \int_{-\tau}^{0} \int_{s}^{0} \sum_{i=1}^{2^m} \alpha_i x_t^T(\xi) A_i^T R_2 A_i x_t(\xi) \mathrm{d}\xi \mathrm{d}s \\ &+ \int_{-\tau}^{0} \int_{-\tau+s}^{0} x_t^T(\xi) A_d^T R_1 A_d x_t(\xi) \mathrm{d}\xi \mathrm{d}s \\ &+ \int_{-\tau}^{0} \int_{t+s}^{t} f^T(\xi) E^T W_1 E f(\xi) \mathrm{d}\xi \mathrm{d}s \end{aligned} \tag{16}$$

where P, R_1, R_2, W_1 are symmetric positive-definite matrices with appropriate dimensions, A_d, A_i, E are matrices with appropriate dimensions, $\sum_{i=1}^{2m} \alpha_i = 1$ and $\sup_{t\in[0,\infty)} \|f(t)\| = \sigma_f$. The derivative of $V(t, x_t)$ is

$$\dot{V}(t, x_t) \leqslant -\lambda_{\min}(Q) \|x(t)\|^2 + \left\|E^T W_2 E\right\| \sigma_f^2 + \tau \left\|E^T W_1 E\right\| \sigma_f^2 \tag{17}$$

where Q, W_2 are symmetric positive-definite matrices with appropriate dimensions. On the basis of Equations (16) and (17), it is pointed out in [5] that only the uniform boundedness of the solutions can be guaranteed.

It is easy to verify that Equations (16) and (17) can be transformed into the forms as Equations (2) and (3), respectively. Then by Theorem 1, we can prove the solutions are uniformly ultimately bounded. This obtained conclusion is stronger than that obtained in [5].

4.3. *Application 3 (less conservative)*

Consider a class of systems with time delays and parameter uncertainties described by [8]

$$\dot{x}(t) = [A + \Delta A(t)] x(t) + \sum_{i=1}^{l} [A_i + \Delta A_i(t)] x(t - \tau_i) + Bu(t) + w(t) \tag{18}$$

with

$$x(s) = \phi(s), s \in \left[-\max_i (\tau_i), 0\right]$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and $w(t) \in \mathbb{R}^n$ is a bounded disturbance vector. The scalars $\tau_i \in \mathbb{R}^+$, $i = 1, \ldots, l$, with $l \in \mathbb{N}$ are delays of the system. $\phi(t)$ is a continuous vector-valued initial function. $A \in \mathbb{R}^{n\times n}$ and $A_i \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, l$, and $B \in \mathbb{R}^{n\times m}$ are known real constant matrices. $\Delta A(t) \in \mathbb{R}^{n\times n}$ and $\Delta A_i(t) \in \mathbb{R}^{n\times n}, i = 1, \ldots, l$, are time-varying matrices representing uncertainties in the system parameters. It is assumed that the right-hand side of Equation (18) is continuous and satisfies enough smoothness conditions to ensure the existence and uniqueness of the solution through every initial condition $\phi(t)$. To make the resulting closed-loop system converge to a ball with a certain convergence rate uniformly exponentially, an adaptive robust controller proposed in [8] is designed as

$$\begin{aligned} u(t) &= \left(YP^{-1} - y(t) B^T P^{-1}\right) x(t) \\ \dot{y}(t) &= \left\|x^T(t) P^{-1} B\right\|^2 - by(t) \end{aligned} \tag{19}$$

DOI: 10.1002/rnc

where $b \in \mathbb{R}^+$, $y(t) \in \mathbb{R}$, $P \in \mathbb{R}^{n \times n}$ is a symmetric positive-definite matrix, and $Y \in \mathbb{R}^{m \times n}$ is a constant matrix. For stability analysis, the Lyapunov–Krasovskii functional in [8] is designed as

$$V(t, x_t, y) = x^T(t) P^{-1} x(t) + \|y(t)\|^2 + \sum_{i=1}^{l} \int_{-\tau_i}^{0} x_t^T(s) P^{-1} Q_i P^{-1} x_t(s) \, ds \tag{20}$$

where $Q_i, i = 1, \ldots, l$ are symmetric positive-definite matrices with appropriate dimensions. The derivative of $V(t, x_t, y)$ is

$$\dot{V}(t, x_t, y) \leqslant -\mu \|x(t)\|^2 - \mu \|y(t)\|^2 + \sigma \tag{21}$$

where $\mu, \sigma \in \mathbb{R}^+$. Because $x(t)$ and $y(t)$ are the states of the closed-loop systems (18) and (19) considered in [8], the closed-loop system can be represented in the form of equation (1).

- *Existing method.* On the basis of Equations (20) and (21), after using a transformation and inequality techniques, the authors in [8] finally obtain

$$\left\| \begin{bmatrix} x^T(t) & y^T(t) \end{bmatrix}^T \right\| \leqslant b_{u,1} + k_{u,1} e^{-\lambda_{u,1} t} \tag{22}$$

where $b_{u,1} = [\sigma/\theta \min(\|P\|, 1)]^{1/2}$ (note that $\|P\| = \lambda_{\min}(P^{-1})$) and $k_{u,1}, \lambda_{u,1} \in \mathbb{R}^+$. The scalar θ is the unique root to the following equation

$$\theta\eta + \theta\eta\tilde{\tau} e^{\theta\tilde{\tau}} - \mu = 0 \tag{23}$$

where $\eta = \max\left(\|P^{-1}\|, \sum_{i=1}^{l} \|P^{-1} Q_i P^{-1}\|, 1\right)$ and $\tilde{\tau} = \max_{1 \leqslant i \leqslant l}(\tau_i)$. The value $b_{u,1} + \epsilon_1$ is the ultimate bound, where $\epsilon_1 \in \mathbb{R}^+$ is an arbitrarily small number. It is shown that the ultimate boundedness criterion in [6] is improved in [8]. However, the parameter θ is not given explicitly. Accordingly, $b_{u,1}$ is not given explicitly.
- *New method.* Now, we will use Theorem 1 to give a more explicit and tighter ultimate bound. Define $z = \begin{bmatrix} x^T & y^T \end{bmatrix}^T$ and $V(t, z_t) = V(t, x_t, y)$. Equation (20) can be transformed into Equation (2) with $\gamma_1 = \min(\|P\|, 1)$, $\gamma_2 = \max\left(\|P^{-1}\|, 1\right)$, $\rho_i = \|P^{-1} Q_i P^{-1}\|$ and $N_1 = l$. Inequality (21) can be written as Equation (3) with $\gamma_3 = \mu$. By Theorem 1, we can obtain

$$\|x(t)\| \leqslant b_{u,2} + k_{u,2} e^{-\lambda_{u,2} t} \tag{24}$$

where $k_{u,2}, \lambda_{u,2} \in \mathbb{R}^+$, $b_{u,2} = [\sigma/\theta' \min(\|P\|, 1)]^{1/2}$ and $\theta' = \mu/\eta'$, $\eta' = \max\left(\|P^{-1}\|, 1\right) + \sum_{i=1}^{l} \|P^{-1} Q_i P^{-1}\| \tau_i$. The value $b_{u,2} + \epsilon_2$ is the ultimate bound, where $b_{u,2}$ is given explicitly and $\epsilon_2 \in \mathbb{R}^+$ is an arbitrarily small number.
- *Comparison.* Next, the following proposition will demonstrate that $b_{u,2}$ proposed in Equation (24) is less conservative than $b_{u,1}$ proposed in Equation (22), namely $b_{u,2} + \epsilon_2 < b_{u,1} + \epsilon_1$.

Proposition 1
$b_{u,2} < b_{u,1}$ for all $\sigma \in \mathbb{R}^+$ and all $\mu, \tilde{\tau} \in \mathbb{R}^+$ satisfying Equation (23).

Proof
From Equation (23), we have $1/\theta = \eta\left(1 + \tilde{\tau} e^{\tilde{\tau}\theta}\right)/\mu > 0$, where $\eta, \mu, \tilde{\tau}$ are defined in Equation (23). Because $\theta > 0$, it gives $\tilde{\tau} e^{\tilde{\tau}\theta} > \tilde{\tau}$ that implies $1/\theta > \eta(1 + \tilde{\tau})/\mu$. Then, by the definition of $b_{u,1}$, it follows that $[(1 + \tilde{\tau})\eta\sigma/\mu \min(\|P\|, 1)]^{1/2} < b_{u,1}$. On the other hand, by the definition of η', because $\eta' \leqslant \max\left(\|P^{-1}\|, 1\right) + \tilde{\tau} \sum_{i=1}^{l} \|P^{-1} Q_i P^{-1}\| \leqslant (1 + \tilde{\tau})\eta$, where $\eta, \tilde{\tau}$ are defined in Equation (23), we obtain $b_{u,2} \leqslant [(1 + \tilde{\tau})\eta\sigma/\mu \min(\|P\|, 1)]^{1/2}$. Therefore, we can conclude this proof. $\square$

Let $\epsilon_1 = \epsilon_2$, then the proposed ultimate bound $b_{u,2} + \epsilon_2$ is more explicit and tighter than the ultimate bound $b_{u,1} + \epsilon_1$ proposed in [8].

Remark 5

$b_{u,1} + \epsilon_1$ and $b_{u,2} + \epsilon_2$ are the ultimate bound, whereas $b_{u,1} + k_{u,1}$ and $b_{u,2} + k_{u,2}$ are the bound. The terms $k_{u,1}e^{-\lambda_{u,1}t}$ and $k_{u,2}e^{-\lambda_{u,2}t}$ will vanish as $t \to \infty$. The proposed method only can prove that the ultimate bound should be tighter than that claimed in [8] but will not change the closed-loop system. Therefore, the numerical simulations are the same as that in [8].

5. CONCLUSIONS

A new method is proposed to determine the ultimate bounds and the convergence rates for perturbed time-delay systems when the Lyapunov–Krasovskii functionals and their derivatives are available. From the applications, it is clear that the proposed method is more concise, more widely applicable, and the obtained results are less conservative compared with existing methods. It is expected that the proposed method can be applied to more applications, for Lyapunov–Krasovskii functionals therein can be chosen with flexibility.

ACKNOWLEDGEMENTS

This work was supported by the National Natural Science Foundation of China (61104012), the 973 Program (2010CB327904), and the Foundation of BUAA for Young Teachers (YWF-11-03-Q-013).

REFERENCES

1. Kofman E, Haimovich H, Seron MM. A systematic method to obtain ultimate bounds for perturbed systems. *International Journal of Control* 2007; **80**(2):167–178.
2. Kofman E, Seron MM, Haimovich H. Control design with guaranteed ultimate bound for perturbed systems. *Automatica* 2008; **44**(7):1815–1821.
3. Oucheriah S. Robust tracking and model following of uncertain dynamic delay systems by memoryless linear controllers. *IEEE Transactions on Automatic Control* 1999; **44**(7):1473–1577.
4. Oucheriah S. Adaptive robust control of a class of dynamic delay systems with unknown uncertainty. *International Journal of Adaptive Control and Signal Processing* 2001; **15**(1):53–63.
5. Oucheriah S. Synthesis of controllers for time-delay systems subject to actuator saturation and disturbance. *Journal of Dynamic Systems, Measurement, and Control* 2003; **125**(2):244–249.
6. Zheng F, Wang Q-G, Lee TH. Adaptive robust control of uncertain time delay systems. *Automatica* 2005; **41**(8):1375–1383.
7. Oucheriah S. Robust exponential convergence of a class of linear delayed systems with bounded controllers and disturbances. *Automatica* 2006; **42**(11):1863–1867.
8. Xu S, Feng G. Further results on adaptive robust control of uncertain time-delay systems. *IET Control Theory and Applications* 2008; **2**(5):402–408.
9. Sontag ED. Input to state stability: basic concepts and results. In *Nonlinear and Optimal Control Theory*, Nistri P, Stefani G (eds). Springer-Verlag: Berlin, Germany, 2007; 166–220.
10. Teel AR. Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Transaction on Automatic Control* 1998; **43**(7):960–964.
11. Pepe P, Jiang Z-P. A Lyapunov–Krasovskii methodology for ISS and iISS of time-delay systems. *Systems & Control Letters* 2006; **55**(12):1006–1014.
12. Pepe P. The problem of the absolute continuity for Lyapunov–Krasovskii functionals. *IEEE Transaction on Automatic Control* 2007; **52**(5):953–957.