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A stability theorem of the direct Lyapunov's method for neutral-type systems in a critical case

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A new stability theorem of the direct Lyapunov's method is proposed for neutral-type systems. The main contribution of the proposed theorem is to remove the condition that the $\mathcal{D}$ operator is stable. In order to demonstrate the effectiveness, the proposed theorem is used to determine the stability of a neutral-type system in a critical case, i.e. the dominant eigenvalues of the principal neutral term (matrix D in Introduction) lie on the unit circle. This is difficult or infeasible in previous studies.

Keywords: neutral-type systems; stability; critical case

1. Introduction

To the best of the authors' knowledge the existing stability theorems of the direct Lyapunov's method for neutral-type systems all require the condition that the $\mathcal{D}$ operator is stable (Hale 1977, p. 297; Kolmanovskii and Myshkis 1999, pp. 329–337). More concretely, consider a $\mathcal{D}$ operator defined as $\mathcal{D}x_t = x(t) - Dx(t-\tau)$, $x_t \triangleq x(t+\theta)$, $\theta \in [-\tau, 0]$. The stable $\mathcal{D}$ operator implies that spectral radius $\rho(D) < 1$, i.e. matrix D is Schur stable. On the other hand, the case $\rho(D) \geq 1$ means that the $\mathcal{D}$ operator is not stable. Therefore, the existing stability theorems cannot be applied to the case $\rho(D) = 1$, namely the critical case.

Neutral-type systems in the critical case are often related to a class of repetitive control systems (Hara, Yamamoto, Omata, and Nakano 1988; Quan, Yang, Cai, and Jiang 2009). It is much more complicated to determine the stability of such systems because their characteristic equations may have an infinite sequence of roots with negative real parts approaching zero. In recent years, frequency-domain methods have been applied to investigate stability problem of neutral systems in the critical case (Hale and Verduyn, Lunel 2002, 2003; Rabaha, Sklyarb, and Rezounenkoc 2005). Unfortunately, the frequency-domain stability criteria will become more and more difficult to verify as the dimension of matrix D increases. Moreover, when D has multiple eigenvalues of modulus 1 without Jordan chain, the analysis of non-exponential asymptotic stability is still an 'open problem' according to (Rabaha et al. 2005, pp. 426–427).

Taking these into account, we propose a new stability theorem of the direct Lyapunov's method which does not require the $\mathcal{D}$ operator to be stable for neutral-type systems. The proposed stability theorem is not a trivial extension of the existing stability theorems because the stable $\mathcal{D}$ operator has played an important role in the stability proofs. When the $\mathcal{D}$ operator is stable, $\mathcal{D}x_t$ can play the role of $x(t)$ in retarded-type systems. Consequently, some stability theorems for retarded-type systems can be generalised to neutral-type systems. Moreover, the stable $\mathcal{D}$ operator usually plays a role to prove the boundedness of $\|\dot{x}(t)\|$, which is a very important step to prove the asymptotic stability of neutral-type systems (Hale 1977, pp. 296–297; Kolmanovskii and Myshkis 1999, pp. 330–331, 336–337; Kolmanovskii and Nosov 1986, pp. 157–158). In order to remove the requirement of the stable $\mathcal{D}$ operator, we examine the uniform continuity of $x(t)$ from the definition rather than the boundedness of $\|\dot{x}(t)\|$. In fact, through the investigation in this article, the former plays the same role of the latter in the proofs of existing stability theorems. Using the proposed stability theorem, we derive a stability criterion in terms of a linear matrix inequality (LMI), which can be applied to determine the stability of neutral-type systems in the critical case. This makes the proposed criterion quite feasible with the aid of a computer.

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The notations used in this article are as follows. $\mathbb{R}^n$ is Euclidean space of dimension n. $\lambda_{\min}(X)$, $\lambda_{\max}(X)$ and $\rho(X)$ denote the minimum eigenvalue, the maximum eigenvalue and the spectral radius of positive semi-definite matrix X, respectively. I_n is the identity matrix with dimension n. $\|\cdot\|$ denotes the Euclidean norm or induced matrix norm. In this article, we define the norm $\|\phi\|_W \triangleq \left[\|\phi(0)\|^2 + \int_{-\tau}^{0}\|\dot{\phi}(\theta)\|^2 \mathrm{d}\theta\right]^{\frac{1}{2}}$. $C_{n,\tau}$ denotes the space of continuous n-dimensional vector functions on $[-\tau, 0]$. $L_{n,\tau}$ denotes the space of n-dimensional vector functions ϕ on $[-\tau, 0]$ such that $\int_{-\tau}^{0}\|\phi(\theta)\|^2 \mathrm{d}\theta < \infty$.

2. Problem formulation

Consider the same neutral-type system as in Hale (1977, p. 273):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}x_t = f(t, x_t), \tag{1}$$

where $x_t(\theta) \triangleq x(t+\theta)$, $x_{t_0}(\theta) = \phi(\theta)$, $\phi \in C_{n,\tau}$, $-\tau \le \theta \le 0$, $\tau > 0$, $\Omega \subseteq \mathbb{R} \times C_{n,\tau}$ is open, $f : \Omega \to \mathbb{R}^n$, $\mathcal{D} : C_{n,\tau} \to \mathbb{R}^n$ are given continuous functions with $\mathcal{D}$ atomic at zero. We impose the following assumption on the system (1).

Assumption 1: *The function f and initial condition ϕ are supposed to ensure the solution $x(t_0, \phi)(t)$ through (t_0, ϕ) is unique and, continuously differentiable except maybe at the points $t_0 + k\tau$, $k = 0, 1, 2, \ldots$.*

Remark 1: It is proven in Hale (1977, p. 25, Theorem 7.1) that a class of linear neutral-type systems (1), i.e. $\dot{x}(t) - D\dot{x}(t-\tau) = A_0x(t) - A_1x(t-\tau)$ with $\phi \in C_{n,\tau}$, satisfies Assumption 1.

Before introducing the objective of this article, the definition of the stable operator $\mathcal{D}$ is needed.

Definition 1 (Hale 1977, p. 287, Definition 4.1)**:** $\mathcal{D} : C_{n,\tau} \to \mathbb{R}^n$ is linear, continuous and atomic at 0 and let $C_{\mathcal{D}} = \{\phi \in C_{n,\tau} : \mathcal{D}\phi = 0\}$. The operator $\mathcal{D}$ is said to be stable if the zero solution of the homogeneous 'difference' equation,

$$\mathcal{D}y_t = 0, \quad t \ge t_0, \quad y_{t_0} = \psi \in C_{\mathcal{D}},$$

is uniformly asymptotically stable.

Remark 2: Consider $\mathcal{D}y_t = y(t) - Dy(t-\tau)$ for simplicity. If $\rho(D) < 1$, then $\mathcal{D}$ is stable; but if $\rho(D) = 1$, then $\mathcal{D}$ is not stable. Take $D = I_n$ for example, $\mathcal{D}y_t = 0$ implies $y(t) = y(t-\tau)$. Therefore, the zero solution of the homogeneous 'difference' equation $y(t) = y(t-\tau)$ is bounded and is not uniformly asymptotically stable. Therefore, as far as the authors know, the existing stability theorems cannot be applied to the critical case.

The objective of this article is to derive a stability theorem of the direct Lyapunov's method for the system (1), which does not require the $\mathcal{D}$ operator to be stable. Furthermore, in order to demonstrate the effectiveness, the proposed stability theorem will be used to determine the stability of a special case of (1) in the critical case.

3. A new stability theorem

Before proceeding further, two definitions and a lemma are needed.

Definition 2 (Kolmanovskii and Nosov 1986, pp. 128, 157)**:** The trivial solution of the system (1) is said to be KN-stable if for any $\varepsilon > 0$, there is a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|\phi\|_W < \delta$ implies $\|x(t_0, \phi)(t)\| < \varepsilon$, $t \ge t_0$. The trivial solution of the system (1) is said to be asymptotically KN-stable if the trivial solution is KN-stable, and for any $\varepsilon > 0$, there is a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|\phi\|_W < \delta$ implies $\lim_{t\to\infty}\|x(t_0, \phi)(t)\| = 0$. The trivial solution is said to be globally asymptotically KN-stable if it is KN-stable and $\lim_{t\to\infty}\|x(t_0, \phi)(t)\| = 0$ for any initial condition $\|\phi\|_W < \infty$.

Remark 3: Note that the definitions of stability in Kolmanovskii and Nosov (1986, pp. 128, 157) are slightly different from these proposed in Hale (1977, p. 130). In Hale (1977, p. 130), the initial condition is restricted by $\sup_{\theta\in[-\tau,0]}\|\phi(\theta)\| < \delta$ rather than $\|\phi\|_W < \delta$. The latter depends on the derivative of the initial condition. To distinguish the two definitions, we say that 'stable' in the sense of Kolmanovskii–Nosov is 'KN-stable' here.

Definition 3: Suppose $x : [t_0, \infty) \to \mathbb{R}^n$. We say that $x(t)$ is uniformly continuous on $[t_0, \infty)$ if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|x(t+h) - x(t)\| < \varepsilon$ for all t on $[t_0, \infty)$ with $|h| < \delta$.

Lemma 1: *Under Assumption* 1, *if* $\|x_t(t_0,\phi)\|_W$ *is bounded for all t on* $[t_0, \infty)$, *then* $\|x(t_0,\phi)(t)\|$ *is uniformly continuous on* $[t_0, \infty)$.

Proof: Denote $x = x(t_0, \phi)$ for simplicity in this proof. For a given $h > 0$ without loss of generality, we have

$$\int_{-h}^{0}\|\dot{x}_{t+h}(\theta)\|\mathrm{d}\theta \le \sqrt{h}\left(\int_{-h}^{0}\|\dot{x}_{t+h}(\theta)\|^2\mathrm{d}\theta\right)^{\frac{1}{2}}$$

by using the Cauchy-Schwarz inequality $\langle a, b\rangle \le \|a\| \|b\|$. Since $\|x_t\|_W$ is bounded for all t on $[t_0, \infty)$, denoted by $\|x_t\|_W \le M$, where $M \in \mathbb{R}$ and $M > 0$. Recalling the definition of $\|\cdot\|_W$, we obtain

$$\sup_{t\in[t_0,\infty)}\left(\int_{-\tau}^{0}\|\dot{x}_t(\theta)\|^2\mathrm{d}\theta\right)^{\frac{1}{2}} \le M. \tag{2}$$

Thus

$$\int_{-h}^{0} \left\| \dot{x}_{t+h}(\theta) \right\| \mathrm{d}\theta \leq M\sqrt{N}\sqrt{h},$$

where $N = \lfloor h/\tau \rfloor + 1$ and $\lfloor h/\tau \rfloor$ represents the floor integer of h/τ.

Under Assumption 1, Newton–Leibniz Formula holds as

$$x(t+h) - x(t) = \int_{-h}^{0} \dot{x}_{t+h}(\theta)\mathrm{d}\theta.$$

Then we have

$$\left\| x(t+h) - x(t) \right\| \leq \int_{-h}^{0} \left\| \dot{x}_{t+h}(\theta) \right\| \mathrm{d}\theta \leq M\sqrt{N}\sqrt{h}.$$

For any $\varepsilon > 0$ there exists a $\delta = \varepsilon^2/(M^2N)$ such that $\|x(t+h) - x(t)\| < \varepsilon$ for all t on $[t_0, \infty)$ with $|h| < \delta$. This implies that $x(t_0, \phi)(t)$ is uniformly continuous on $[t_0, \infty)$. □

If $V: \mathbb{R} \times C_{n,\tau} \times L_{n,\tau} \to \mathbb{R}$ is continuous and $x(t_0, \phi)$ is the solution of the system (1) through (t_0, ϕ) we define

$$\dot{V}(t, x_t, \dot{x}_t) = \limsup_{h \to 0^+} \frac{V(t+h, x_{t+h}, \dot{x}_{t+h}) - V(t, x_t, \dot{x}_t)}{h},$$

where $x = x(t_0, \phi)$. The function $\dot{V}(t, x_t, \dot{x}_t)$ is the upper right-hand derivative of $V(t, x_t, \dot{x}_t)$ along the solution of Equation (1).

Theorem 1: *Under Assumption* 1, *suppose* $u(s)$, $v(s)$ *and* $w(s)$ *are continuous, nonnegative, and nondecreasing with* $u(s), v(s) > 0$ *for* $s > 0$, *and* $u(0) = 0$, $v(0) = 0$. *If there is a continuous function* $V: R \times C_{n,\tau} \times L_{n,\tau} \to R$ *such that*

$$u(\|x_t\|_W) \leq V(t, x_t, \dot{x}_t) \leq v(\|x_t\|_W), \tag{3}$$

$$\dot{V}(t, x_t, \dot{x}_t) \leq -w(\|x_t(0)\|), \tag{4}$$

then: (i) *the solution* $x = 0$ *of* (1) *is KN-stable*. (ii) *If* $w(s) > 0$ *for* $s > 0$, *then the solution* $x = 0$ *of* (1) *is asymptotically KN-stable*. (iii) *If* $w(s) > 0$ *for* $s > 0$ *and* $u(s) \to \infty$ *as* $s \to \infty$, *then the solution* $x = 0$ *of* (1) *is globally asymptotically KN-stable*.

Proof: Let $x = x(t_0, \phi)$ and $V(t) = V(t, x_t, \dot{x}_t)$ for simplicity.

(i) First, we need to show that the definition of the continuous function V is appropriate for the solution x_t, i.e. $x_t \in C_{n,\tau}$ and $\dot{x}_t \in L_{n,\tau}$ are both satisfied for all $t \geq t_0$. Since $x_t \in C_{n,\tau}$ is satisfied by Assumption 1, we now pay attention to proving $\dot{x}_t \in L_{n,\tau}$ for all $t \geq t_0$.

For any $\varepsilon > 0$, there is a $\delta = \delta(t_0, \varepsilon)$, $0 < \delta < \varepsilon$, such that $v(\delta) \leq u(\varepsilon)$. For any ϕ such that $\|\phi\|_W \leq \delta$, we claim that $\dot{x}_t \in L_{n,\tau}$ for all $t \geq t_0$. Otherwise, there exists a $t_1 > t_0$ such that $\dot{x}_t \in L_{n,\tau}$ for all $t_0 \leq t < t_1$ but $\dot{x}_{t_1} \notin L_{n,\tau}$. This implies that for any $\varepsilon > 0$, if $\varepsilon > 0$ is sufficiently small, then

$$\left\| x_{t_1 - \epsilon} \right\|_W > \varepsilon. \tag{5}$$

When $\|\phi\|_W \leq \delta$, since $\dot{V}(t) \leq 0$ for all $t_0 \leq t < t_1$, the inequality (3) implies

$$\begin{aligned} u(\|x_t\|_W) &\leq V(t) \leq V(t_0) \leq v(\|\phi\|_W) \\ &\leq v(\delta) \leq u(\varepsilon), \quad t_0 \leq t < t_1. \end{aligned} \tag{6}$$

Therefore

$$\|x_t\|_W \leq \varepsilon, \quad t_0 \leq t < t_1. \tag{7}$$

This contradicts (5). Therefore, $\dot{x}_t \in L_{n,\tau}$ for all $t \geq t_0$ when $\|\phi\|_W \leq \delta$. This implies that the definition of the continuous function V is appropriate.

From (7) and the definition of $\|\cdot\|_W$, we obtain

$$\left\| x(t) \right\| \leq \varepsilon, \quad t \geq t_0.$$

This proves that the trivial solution of the system (1) is KN-stable.

(ii) To prove the assertion of the theorem concerning asymptotic stability, choose a $\delta_0 > 0$ such that $u(\delta) = v(\delta_0)$, where the δ is defined in Proof (i). Then, as previously demonstrated in (6), $\|\phi\|_W < \delta_0$ implies $\|x_t\|_W \leq \delta$ for all $t \geq t_0$. Therefore, we can obtain that $x(t)$ is uniformly continuous on $[t_0, \infty)$ by Lemma 1. Suppose that when $\|\phi\|_W < \delta_0$, a solution x does not tend to zero as $t \to \infty$. Then for some $\varepsilon > 0$ there exists a sequence $\{t_k\}$ such that

$$t_0 + (2k-1)r \leq t_k \leq t_0 + 2kr, \quad k = 1, 2, \ldots$$

and

$$\left\| x(t_k) \right\| \geq \varepsilon, \tag{8}$$

where $r > 0$. By the uniform continuity of $x(t)$, it follows that

$$\left\| x(t) - x(t_k) \right\| < \frac{\varepsilon}{2} \tag{9}$$

on the intervals $t_k - \frac{\varepsilon}{2L} \leq t \leq t_k + \frac{\varepsilon}{2L}, k = 1, 2, \ldots,$ where $L > 0$ is a sufficiently large real number.

Using (8) and (9), we can obtain

$$\begin{aligned} \left\| x(t) \right\| &= \left\| x(t_k) + x(t) - x(t_k) \right\| \\ &\geq \left\| x(t_k) \right\| - \left\| x(t) - x(t_k) \right\| > \frac{\varepsilon}{2} \end{aligned}$$

on the intervals $t_k - \frac{\varepsilon}{2L} \leq t \leq t_k + \frac{\varepsilon}{2L}, k = 1, 2, \ldots.$ Therefore, on the intervals $t_k - \frac{\varepsilon}{2L} \leq t \leq t_k + \frac{\varepsilon}{2L}$, we have $\dot{V}(t) \leq -w(\|x_t(0)\|) \leq -w\left(\frac{\varepsilon}{2}\right) = \gamma < 0.$

Therefore,

$$V(t)-V(t_0)=\int_{t_0}^{t}\dot V(s)\mathrm{d}s \le \sum\nolimits_{t_k\le t}\int_{t_k-\frac{\varepsilon}{2L}}^{t_k+\frac{\varepsilon}{2L}}\dot V(s)\mathrm{d}s \le \frac{\varepsilon}{L}\gamma\cdot N_L(t),$$

where $N_L(t)$ is a number of points t_k such that $t_k<t$. Thus, $V(t)-V(t_0)\le-\infty$ as $t\to\infty$. This contradicts inequality (3). Therefore, a solution x tends to zero as $t\to\infty$ when $\|\phi\|_W<\delta_0$, i.e. the solution $x=0$ of (1) is asymptotically KN-stable.

(iii) For any $\delta_0>0$ such that $\|\phi\|_W<\delta_0$, since $u(s)\to\infty$ as $s\to\infty$, there exists a $\delta<\infty$ such that $u(\delta)=v(\delta_0)$. Then, as previously demonstrated in (6), $\|\phi\|_W<\delta_0$ implies $\|x_t\|_W\le\delta$ for all $t\ge t_0$. The remainder of the proof is the same as that in Proof (ii), so we omit it here. □

Remark 4: Unlike most proofs of existing stability theorems, the proof of Theorem 1 does not require that the $\mathcal{D}$ operator is stable. This implies that Theorem 1 can be applicable to analyse the stability of neutral-type systems in the critical case.

Remark 5: In the proof above, we prove the uniform continuity of $x(t_0,\phi)(t)$ rather than the boundedness of $\|\dot x(t_0,\phi)(t)\|$. Note that the former result is weaker than the latter, i.e. the boundedness of $\|\dot x(t_0,\phi)(t)\|$ implies the uniform continuity of $x(t_0,\phi)(t)$, but not vise versa (Slotine and Li 1991, pp. 123–124).

4. A stability criterion

In order to demonstrate the effectiveness of Theorem 1, the stability of neutral-type systems in the critical case is examined in this section.

For simplicity, consider a simple neutral-type system as follows:

$$\dot x(t)-D\dot x(t-\tau)=A_0x(t)+A_1x(t-\tau) \tag{10}$$

with the initial condition

$$x_{t_0}(\theta)=\varphi(\theta),\quad \theta\in[-\tau,0],$$

where $x(t)\in\mathbb{R}^n$ and D, A_0, $A_1\in\mathbb{R}^{n\times n}$. $\varphi(t)$ is a continuously differentiable smooth vector-valued function representing the initial condition function for an interval of $[-\tau,0]$. We can conclude that the solution $x(t_0,\ \varphi)(t)$ of system (10) satisfies Assumption 1, (Hale 1977, p. 25). For (10), we give a stability criterion as follows.

Theorem 2: *The neutral-type system* (10) *is globally asymptotically KN-stable, if there exist positive definite symmetric matrices* P_1, P_2, Q_1, Q_2, $S\in R^{n\times n}$ *and matrices* K_1, K_2, K_3, $K_4\in R^{n\times n}$ *that satisfy the following LMI*:

$$\Phi+\Gamma S\Gamma^T\le 0, \tag{11}$$

where

$$\Phi=\begin{bmatrix}\Phi_{11} & * & * & *\\ \Phi_{21} & \Phi_{22} & * & *\\ \Phi_{31} & \Phi_{32} & \Phi_{33} & *\\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44}\end{bmatrix},$$

where

$$\Phi_{11}=K_1^TA_0+A_0^TK_1+Q_1,\quad \Phi_{21}=K_2^TA_0+A_1^TK_1,$$
$$\Phi_{22}=K_2^TA_1+A_1^TK_2-Q_1,\quad \Phi_{31}=K_3^TA_0-K_1+P_1,$$
$$\Phi_{32}=K_3^TA_1-K_2,\quad \Phi_{33}=-K_3^T-K_3+Q_2,$$
$$\Phi_{41}=K_4^TA_0+D^TK_1,\quad \Phi_{42}=K_4^TA_1+D^TK_2+P_2,$$
$$\Phi_{43}=-K_4^T+D^TK_3,\quad \Phi_{44}=K_4^TD+D^TK_4-Q_2,$$
$$\Gamma=\begin{bmatrix}I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n}\end{bmatrix}^T.$$

Proof: Choose the candidate Lyapunov functional $V(t)$ to be

$$V(t)=x^T(t)P_1x(t)+x^T(t-\tau)P_2x(t-\tau)+\int_{t-\tau}^{t}x^T(\theta)Q_1x(\theta)\mathrm{d}\theta+\int_{t-\tau}^{t}\dot x^T(\theta)Q_2\dot x(\theta)\mathrm{d}\theta, \tag{12}$$

where P_1, P_2, Q_1, $Q_2\in\mathbb{R}^{n\times n}$ are positive definite symmetric matrices. Taking the derivative of $V(t)$ results in

$$\dot V(t)=2x^T(t)P_1\dot x(t)+2x^T(t-\tau)P_2\dot x(t-\tau)+x^T(t)Q_1x(t)-x^T(t-\tau)Q_1x(t-\tau)+\dot x^T(t)Q_2\dot x(t)-\dot x^T(t-\tau)Q_2\dot x(t-\tau). \tag{13}$$

After introducing a zero term

$$2[K_1x(t)+K_2x(t-\tau)+K_3\dot x(t)+K_4\dot x(t-\tau)]^T\times[A_0x(t)+A_1x(t-\tau)-\dot x(t)+D\dot x(t-\tau)]\equiv 0$$

into (13), we obtain

$$\dot V(t,x_t,\dot x_t)=\chi^T(t)\Omega\chi(t), \tag{14}$$

where K_1, K_2, K_3, $K_4\in\mathbb{R}^{n\times n}$ and $\chi(t)=\begin{bmatrix}x^T(t) & x^T(t-\tau) & \dot x^T(t) & \dot x^T(t-\tau)\end{bmatrix}^T$. By using (11), we have $\Phi\le-\Gamma S\Gamma^T$. Thus (14) becomes

$$\dot V(t,x_t,\dot x_t)\le -x^T(t)Sx(t), \tag{15}$$

where $x(t)=\Gamma^T\chi(t)$ is utilised.

The remainder of the proof is to examine whether (12) and (15) have the properties (3) and (4), respectively. Based on (15), inequality (4) holds with $w(s)=\lambda_{\min}(S)s^2$. Based on (12), the left-hand side of inequality (4) holds with $u(s)=\alpha s^2$, where $\alpha = \min[\lambda_{\min}(P_1), \lambda_{\min}(Q_1)]$. The next step is to examine the right-hand side of inequality (4). Since $\varphi(t)$ is continuously differentiable, the solution $x(t_0, \varphi)(t)$ is continuously differentiable except maybe at the points $k\tau$, $k = 0, 1, 2, \ldots$ (Hale 1977, p. 25, Theorem 7.1). Then, by Newton–Leibniz Formula, it follows that

$$x(\theta) = x(t) - \int_\theta^t \dot{x}(\xi)\mathrm{d}\xi$$

for $\theta \in [t-\tau, t]$. Based on the above equation, we have

$$\begin{aligned}\int_{t-\tau}^t \|x(\theta)\|^2 \mathrm{d}\theta &= \int_{t-\tau}^t \left\| x(t) - \int_\theta^t \dot{x}(\xi)\mathrm{d}\xi \right\|^2 \mathrm{d}\theta \\ &\le 2\tau\|x(t)\|^2 + 2\int_{t-\tau}^t \left(\int_\theta^t \|\dot{x}(\xi)\|\mathrm{d}\xi \right)^2 \mathrm{d}\theta.\end{aligned} \tag{16}$$

Using the Cauchy–Schwarz inequality $\langle a, b\rangle^2 \le \langle a, a\rangle \langle b, b\rangle$, we obtain

$$\left(\int_\theta^t \|\dot{x}(\xi)\|\mathrm{d}\xi \right)^2 \le (t-\theta)\int_\theta^t \|\dot{x}(\xi)\|^2 \mathrm{d}\xi \le \tau \int_{t-\tau}^t \|\dot{x}(\xi)\|^2 \mathrm{d}\xi.$$

Substituting the inequality above into (16) results in

$$\int_{t-\tau}^t \|x(\theta)\|^2 \mathrm{d}\theta \le 2\tau\|x(t)\|^2 + 2\tau^2 \int_{t-\tau}^t \|\dot{x}(\theta)\|^2 \mathrm{d}\theta. \tag{17}$$

Similarly, we have

$$\|x(t-\tau)\|^2 \le 2\|x(t)\|^2 + 2\tau \int_{t-\tau}^t \|\dot{x}(\theta)\|^2 \mathrm{d}\theta. \tag{18}$$

By using (17) and (18), the right-hand side of inequality (4) holds with $v(s) = \beta s^2$, where

$$\begin{aligned}\beta = \max(&\lambda_{\max}(P_1) + 2\lambda_{\max}(P_2) + 2\tau\lambda_{\max}(Q_1), \\ &2\tau\lambda_{\max}(P_2) + 2\tau^2\lambda_{\max}(Q_1) + \lambda_{\max}(Q_2)).\end{aligned}$$

Therefore, the solution $x(t_0, \varphi)$ of system (10) is globally asymptotically KN-stable. □

Remark 6: Theorem 2 does not require $\rho(D) < 1$. Therefore, Theorem 2 can be applicable to determine the stability of a neutral-type system in a critical case.

Example 1: Consider the simple neutral-type system (10) with

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} -2 & -1 \\ 0.1 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix}.$$

In this case, $\rho(D) = 1$. Solving the LMI (11) results in (rounded to two decimals)

$$P_1 = \begin{bmatrix} 6.25 & 7.92 \\ 7.92 & 16.74 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.34 & 0.46 \\ 0.46 & 1.22 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 6.22 & 6.91 \\ 6.91 & 12.37 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 3.35 & 4.57 \\ 4.57 & 12.2 \end{bmatrix},$$

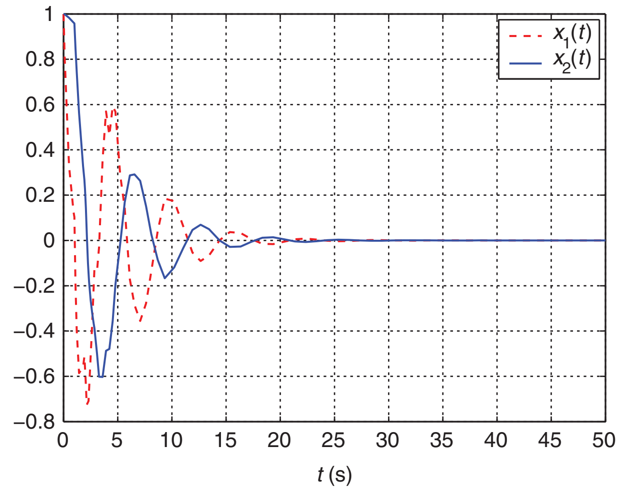

Figure 1. The evolution of a neutral-type system in the critical case.

$$K_1 = \begin{bmatrix} 4.33 & 3.66 \\ -0.13 & 5.95 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.04 & -0.17 \\ -0.22 & -0.26 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} 3.02 & 2.78 \\ 1.51 & 8.36 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 0.34 & 1.79 \\ 3.06 & 3.82 \end{bmatrix}, \quad S = 0.1I_2.$$

Therefore, by Theorem 2, the solution $x(t_0, \varphi)$ of system (10) is globally asymptotically KN-stable. To illustrate the system evolution for $\tau = 1$, let $\varphi(\theta) = [1; 1]$, $\theta \in [-1, 0]$. The evolution of states $x_1(t)$ and $x_2(t)$ is shown in Figure 1. From Figure 1, the solution $x(t_0, \varphi)$ of system (10) tends to zero.

Remark 7: In Example 1, D in the linear neutral-type system (10) has multiple eigenvalues of Modulus 1 without Jordan chain. The stability analysis of the neutral-type system in Example 1 is still an 'open problem' according to Rabaha et al. (2005, pp. 426–427). However, Theorem 2 can be used to determine the stability of such a system.

5. Conclusions

A new stability theorem of the direct Lyapunov's method is proposed to determine the stability of neutral-type systems. Unlike most existing stability theorems, the proposed stability theorem does not require the $\mathcal{D}$ operator to be stable. Using the new stability theorem, we derive a new stability criterion, which can be used to determine the stability of neutral-type systems in the critical case, furthermore, including the case that the principal neutral term D has multiple eigenvalues of module 1 without Jordan chain. This is difficult or infeasible in previous studies.

An illustrative example is presented to demonstrate the effectiveness of the proposed stability theorem.

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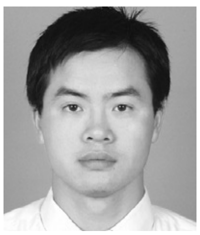

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