



Additive-state-decomposition-based tracking control for TORA benchmark



Quan Quan^{a,b,*}, Kai-Yuan Cai^{a,b}

^a Department of Automatic Control, Beihang University (Beijing University of Aeronautics and Astronautics), Beijing 100191, China

^b State Key Laboratory of Virtual Reality Technology and Systems, Beihang University, Beijing 100191, China

ARTICLE INFO

Article history:

Received 6 February 2012

Received in revised form

16 April 2013

Accepted 19 April 2013

Handling Editor: D.J. Wagg

Available online 5 June 2013

ABSTRACT

In this paper, a new control scheme, called additive-state-decomposition-based tracking control, is proposed to solve the tracking (rejection) problem for rotational position of the translational oscillator with a rotational actuator (TORA, a nonlinear nonminimum phase system). By the additive state decomposition, the tracking (rejection) task for the considered nonlinear system is decomposed into two independent subtasks: a tracking (rejection) subtask for a linear time invariant (LTI) system, leaving a stabilization subtask for a derived nonlinear system. By the decomposition, the proposed tracking control scheme avoids solving regulation equations and can tackle the tracking (rejection) problem in the presence of any external signal (except for the frequencies at ± 1) generated by a marginally stable autonomous LTI system. To demonstrate the effectiveness, numerical simulation is given.

© 2013 Elsevier Ltd. All rights reserved.

1. Introduction

The tracking (rejection) problem for a nonlinear benchmark system called translational oscillator with a rotational actuator (TORA) and also known as rotational–translational actuator (RTAC) has received a considerable amount of attention these years [1–11]. Due to its special nonminimum phase and nonlinearity, the TORA is often of independent interest to serve as a benchmark for different nonlinear control methods. Some results were presented to concern the tracking (rejection) problem for general external signals [2,4,5]. However, the proposed control methods cannot achieve asymptotic disturbance rejection. By taking this into account, the nonlinear output regulation theory was applied to track (reject) external signals generated by an autonomous system. The considered external signals are often constant or periodic due to some periodic operations and vibrations, which can often be modeled as an autonomous system approximately. In this case, asymptotic disturbance rejection can be achieved. With different measurement, the tracking (rejection) problem for translational displacement of the TORA was investigated [6–8]. Readers can refer to [8] for details. Based on the same benchmark system, some other work was also presented to concern the tracking (rejection) problem for rotational position [9,10]. In this problem, the rotational actuator may be considered as a rotational antenna, which is required to point a given direction. For the two types of tracking (rejection) problems, regulator equations have to be solved and then the resulting solutions will be further used in the controller design. However, the difficulty of constructing and solving regulator equations will increase as the complexity of external signals increases.

* Corresponding author at: Department of Automatic Control, Beihang University (Beijing University of Aeronautics and Astronautics), Beijing 100191, China. Tel.: +86 13 81017 1104; fax: +86 10 8233 8414.

E-mail address: qq_buaa@buaa.edu.cn (Q. Quan).

URL: <http://quanquan.buaa.edu.cn/> (Q. Quan).

To overcome these drawbacks above, the tracking (rejection) problem for rotational position of the TORA as [9,10] is revisited by a new control scheme called additive-state-decomposition-based tracking control, which is based on the additive state decomposition.¹ The proposed additive state decomposition is a new type of decomposition different from the lower-order subsystem decomposition. Concretely, taking the system $\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$ for example, it is decomposed into two subsystems: $\dot{\mathbf{x}}_1(t) = \mathbf{f}_1(t, \mathbf{x}_1, \mathbf{x}_2)$ and $\dot{\mathbf{x}}_2(t) = \mathbf{f}_2(t, \mathbf{x}_1, \mathbf{x}_2)$, where $\mathbf{x}_1 \in \mathbb{R}^{n_1}$ and $\mathbf{x}_2 \in \mathbb{R}^{n_2}$, respectively. The lower-order subsystem decomposition satisfies

$$n = n_1 + n_2 \quad \text{and} \quad \mathbf{x} = \mathbf{x}_1 \oplus \mathbf{x}_2.$$

By contrast, the proposed additive state decomposition satisfies

$$n = n_1 = n_2 \quad \text{and} \quad \mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2.$$

In our opinion, lower-order subsystem decomposition aims to reduce the complexity of the system itself, while the additive state decomposition emphasizes the reduction of the complexity of tasks for the system.

By following the philosophy above, the original tracking (rejection) task is “additively” decomposed into two independent subtasks: an output tracking (rejection) subtask for a linear time invariant (LTI) system and a state stabilization subtask for a derived nonlinear system. Since tracking (rejection) subtask only needs to be achieved on an LTI system, the complexity of external signals can be handled easier by the transfer function method. It is proved that the designed controller can tackle the tracking (rejection) problem for rotational position of the TORA in the presence of any external signal (except for the frequency at ± 1) generated by a marginally stable autonomous LTI system.

This paper is organized as follows. In Section 2, the problem is formulated and the additive state decomposition is recalled briefly first. In Section 3, an observer is proposed to compensate for nonlinearity; then the resulting system is “additively” decomposed into two subsystems; sequently, controllers are designed for them. In Section 4, numerical simulation is given. Section 5 further discusses the proposed method. Section 6 concludes this paper.

2. Nonlinear benchmark problem and additive state decomposition

2.1. Nonlinear benchmark problem

As shown in Fig. 1, the TORA system consists of a cart attached to a wall with a spring. The cart is affected by a disturbance force F . An unbalanced point mass rotates around the axis in the center of the cart, which is actuated by a control torque N . The translational displacement of the cart is denoted by x_c and the rotational position of the unbalanced point mass is denoted by θ .

For simplicity, after normalization and transformation, the TORA system is described by the following state-space representation [1]:

$$\dot{x}_1 = x_2 \tag{1a}$$

$$\dot{x}_2 = -x_1 + \varepsilon \sin x_3 + F_d \tag{1b}$$

$$\dot{x}_3 = x_4 \tag{1c}$$

$$\dot{x}_4 = u - \frac{\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} F_d, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{1d}$$

where $0 < \varepsilon < 1$, $\mathbf{x} = [x_1 \ x_2 \ x_3 \ x_4]^T \in \mathbb{R}^4$, $x_3 = \theta$, $F_d \in \mathbb{R}$ is the unknown dimensionless disturbance, $u \in \mathbb{R}$ is the dimensionless control torque. In this paper, the tracking (rejection) problem for rotational position of the TORA as [9,10] is revisited. Concretely, for system (1), it is to design a controller u such that the output $y(t) = x_3(t) \rightarrow r$ as $t \rightarrow \infty$, meanwhile keeping the other states bounded, where $r \in (-\pi/2, \pi/2)$ is a known constant. Obviously, this is a nonlinear nonminimum phase tracking problem, or say a nonlinear weakly minimum phase tracking problem. For system (1), the following assumptions are imposed.

Assumption 1. The state $\mathbf{x}$ can be obtained.

Assumption 2. The disturbance $F_d \in \mathbb{R}$ is generated by an autonomous LTI system

$$\dot{\mathbf{w}} = \mathbf{S}\mathbf{w}, \quad F_d = \mathbf{C}_d^T \mathbf{w} \tag{2}$$

where $\mathbf{S} = -\mathbf{S}^T \in \mathbb{R}^{m \times m}$, $\mathbf{C}_d \in \mathbb{R}^m$ are constant, $\mathbf{w} \in \mathbb{R}^m$, and the pair $(\mathbf{C}_d^T, \mathbf{S})$ is observable.

Remark 1. If all eigenvalues of $\mathbf{S}$ have zero real part, then, in suitable coordinates, the matrix $\mathbf{S}$ can always be written as a skew-symmetric matrix. The matrix $\mathbf{S}$ in previous literature on the output regulation problem is often chosen in a simple form $\mathbf{S} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, where ω is a positive real [6–10]. In such a case, F_d is in the form as $\sin(\pm \omega t)$ and the solution to the regulator equation is easier to obtain. However, this becomes difficult when $\mathbf{S}$ is complicated.

¹ In this paper we have replaced the term “additive decomposition” in [12] with the more descriptive term “additive state decomposition”.

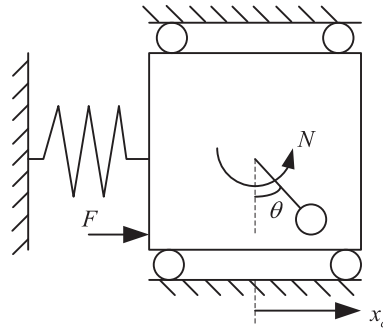


Fig. 1. TORA system configuration [1].

2.2. Additive state decomposition

In order to make the paper self-contained, the additive state decomposition [12] is recalled here briefly. Consider the following “original” system:

$$\mathbf{f}(t, \dot{\mathbf{x}}, \mathbf{x}) = 0, \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (3)$$

where $\mathbf{x} \in \mathbb{R}^n$. We first bring in a “primary” system having the same dimension as (3), according to

$$\mathbf{f}_p(t, \dot{\mathbf{x}}_p, \mathbf{x}_p) = 0, \quad \mathbf{x}_p(0) = \mathbf{x}_{p,0} \quad (4)$$

where $\mathbf{x}_p \in \mathbb{R}^n$. From the original system (3) and the primary system (4) we derive the following “secondary” system:

$$\mathbf{f}(t, \dot{\mathbf{x}}, \mathbf{x}) - \mathbf{f}_p(t, \dot{\mathbf{x}}_p, \mathbf{x}_p) = 0, \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (5)$$

where $\mathbf{x}_p \in \mathbb{R}^n$ is given by the primary system (4). Define a new variable $\mathbf{x}_s \in \mathbb{R}^n$ as follows:

$$\mathbf{x}_s \triangleq \mathbf{x} - \mathbf{x}_p. \quad (6)$$

Then the secondary system (5) can be further written as follows:

$$\mathbf{f}(t, \dot{\mathbf{x}}_p + \dot{\mathbf{x}}_s, \mathbf{x}_p + \mathbf{x}_s) - \mathbf{f}_p(t, \dot{\mathbf{x}}_p, \mathbf{x}_p) = 0, \quad \mathbf{x}_s(0) = \mathbf{x}_0 - \mathbf{x}_{p,0}. \quad (7)$$

From the definition (6), we have

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_s(t), \quad t \geq 0. \quad (8)$$

Remark 2. By the additive state decomposition, the system (3) is decomposed into two subsystems with the same dimension as the original system. In this sense our decomposition is “additive”. In addition, this decomposition is with respect to state. So, we call it “additive state decomposition”.

As a special case of (3), a class of differential dynamic systems is considered as follows:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{h}(t, \mathbf{x}) \end{aligned} \quad (9)$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$. Two systems, denoted by the primary system and (derived) secondary system, respectively, are defined as follows:

$$\begin{aligned} \dot{\mathbf{x}}_p &= \mathbf{f}_p(t, \mathbf{x}_p), \quad \mathbf{x}_p(0) = \mathbf{x}_{p,0} \\ \mathbf{y}_p &= \mathbf{h}_p(t, \mathbf{x}_p) \end{aligned} \quad (10)$$

and

$$\begin{aligned} \dot{\mathbf{x}}_s &= \mathbf{f}(t, \mathbf{x}_p + \mathbf{x}_s) - \mathbf{f}_p(t, \mathbf{x}_p), \quad \mathbf{x}_s(0) = \mathbf{x}_0 - \mathbf{x}_{p,0} \\ \mathbf{y}_s &= \mathbf{h}(t, \mathbf{x}_p + \mathbf{x}_s) - \mathbf{h}_p(t, \mathbf{x}_p) \end{aligned} \quad (11)$$

where $\mathbf{x}_s \triangleq \mathbf{x} - \mathbf{x}_p$ and $\mathbf{y}_s \triangleq \mathbf{y} - \mathbf{y}_p$. The secondary system (11) is determined by the original system (9) and the primary system (10). From the definition, we have

$$\mathbf{x}(t) = \mathbf{x}_p(t) + \mathbf{x}_s(t), \quad \mathbf{y}(t) = \mathbf{y}_p(t) + \mathbf{y}_s(t), \quad t \geq 0. \quad (12)$$

3. Additive-state-decomposition-based tracking control

In this section, an observer is first proposed to compensate for nonlinearity. After the compensation, the resulting nonlinear nonminimum phase tracking system is decomposed into two systems by the additive state decomposition: an LTI system including all external signals as the primary system, leaving the secondary system with a zero equilibrium point. Therefore, the tracking problem for the original system is correspondingly decomposed into two subproblems by the additive state decomposition: a tracking problem for the LTI “primary” system and a stabilization problem for the secondary system. Obviously, the two subproblems are easier than the original one. Therefore, the original tracking problem is simplified. The structure of the closed-loop system is shown in Fig. 2.

3.1. Nonlinearity compensation

First, in order to estimate the term $(\varepsilon \cos x_3 / (1 - \varepsilon^2 \cos^2 x_3))F_d$, an observer is designed, which is stated in Theorem 1.

Theorem 1. Under Assumptions 1 and 2, let the observer be designed for system (1) as follows:

$$\begin{aligned}\dot{\hat{\mathbf{w}}} &= \mathbf{S}\hat{\mathbf{w}} + l_1 \frac{\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} \mathbf{C}_d(\hat{x}_4 - x_4) \\ \dot{\hat{x}}_4 &= -l_2(\hat{x}_4 - x_4) - l_1 \frac{\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} \mathbf{C}_d^T \hat{\mathbf{w}} + u \\ \hat{F}_d &= l_1 \mathbf{C}_d^T \hat{\mathbf{w}}, \quad \hat{\mathbf{w}}(0) = \mathbf{0}, \quad \hat{x}_4(0) = 0\end{aligned}\quad (13)$$

where $l_1, l_2 > 0$. Then $\lim_{t \rightarrow \infty} (\varepsilon \cos x_3 / (1 - \varepsilon^2 \cos^2 x_3)) \hat{F}_d(t) = 0$, where $\tilde{F}_d \triangleq \hat{F}_d - F_d$.

Proof. See Appendix A.

By using the observer (13), the controller u in (1) is designed as follows:

$$u = \mathbf{K}^T \mathbf{x} + v + \frac{\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} \hat{F}_d$$

where $\mathbf{K} \in \mathbb{R}^4$ and $v \in \mathbb{R}$ will be specified later. Then the system (1) becomes

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon \sin x_3 + F_d \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \mathbf{K}^T \mathbf{x} + v + \frac{\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} \hat{F}_d, \quad \mathbf{x}(0) = \mathbf{x}_0.\end{aligned}\quad (14)$$

3.2. Additive state decomposition of original system

Introduce a zero term $\varepsilon \mathbf{D}(\mathbf{C} + a\mathbf{B})^T \mathbf{x} - \varepsilon \mathbf{D}(\mathbf{y} + a\mathbf{y}) = 0$ into the system (14), where $a > 0$, $\mathbf{B} = [0 \ 0 \ 0 \ 1]^T$, $\mathbf{C} = [0 \ 0 \ 1 \ 0]^T$, $\mathbf{D} = [0 \ 1 \ 0 \ 0]^T$ and $\mathbf{B}^T \mathbf{x} = \dot{y}$. Then the system (14) becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}v + \phi(y, \dot{y}) + \mathbf{D}F_d + \varphi$$

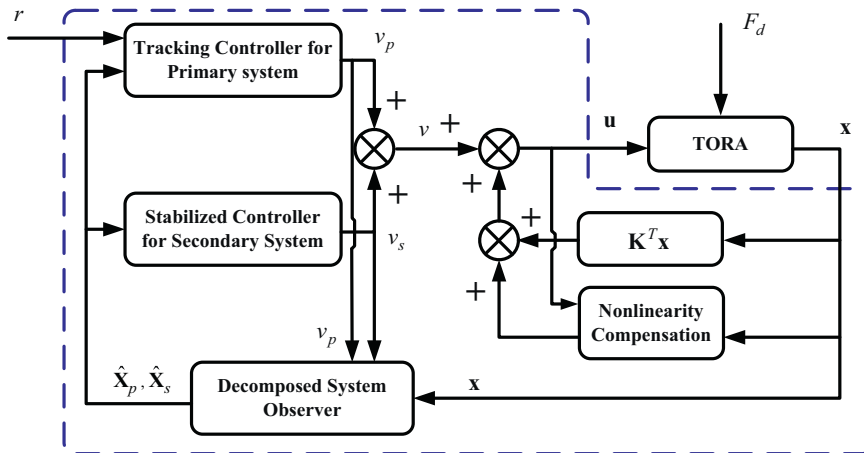


Fig. 2. Structure of the closed-loop system.

$$\dot{\mathbf{y}} = \mathbf{C}^T \mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (15)$$

where

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{B}\mathbf{K}^T + \varepsilon \mathbf{D}(\mathbf{C} + a\mathbf{B})^T, \quad \mathbf{A}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \phi(y, \dot{y}) = \begin{bmatrix} 0 \\ \varepsilon \sin y - \varepsilon(y + a\dot{y}) \\ 0 \\ 0 \end{bmatrix}, \quad \varphi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} \tilde{F}_d \end{bmatrix}. \quad (16)$$

The additive state decomposition is ready to apply to the system (15), for which the primary system is chosen to be an LTI system as follows:

$$\begin{aligned} \dot{\mathbf{x}}_p &= \mathbf{A}\mathbf{x}_p + \mathbf{B}v_p + \mathbf{d} + \varphi \\ y_p &= \mathbf{C}^T \mathbf{x}_p, \quad \mathbf{x}_p(0) = \mathbf{x}_0 \end{aligned} \quad (17)$$

where $\mathbf{d} = \phi(r, 0) + \mathbf{D}\tilde{F}_d$. Then, according to the rule (11), the secondary system is derived from the original system (15) and the primary system (17) as follows:

$$\begin{aligned} \dot{\mathbf{x}}_s &= \mathbf{A}\mathbf{x}_s + \mathbf{B}v_s + \phi(y_p + y_s, \dot{y}_p + \dot{y}_s) - \phi(r, 0) \\ y_s &= \mathbf{C}^T \mathbf{x}_s, \quad \mathbf{x}_s(0) = \mathbf{0} \end{aligned} \quad (18)$$

where $v_s = v - v_p$. According to (12), we have

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_s \quad \text{and} \quad y = y_p + y_s. \quad (19)$$

Remark 3. The pair $(\mathbf{A}_0, \mathbf{B})$ is uncontrollable, while the pair $(\mathbf{A}_0 + \varepsilon \mathbf{D}(\mathbf{C} + a\mathbf{B})^T, \mathbf{B})$ is controllable. Therefore, there always exists a vector $\mathbf{K}$ such that $\mathbf{A} = \mathbf{A}_0 + \mathbf{B}\mathbf{K}^T + \varepsilon \mathbf{D}(\mathbf{C} + a\mathbf{B})^T$ is a stable matrix.

Remark 4. If $y_p \equiv r$ and $\dot{y}_p \equiv 0$, then $(\mathbf{x}_s, v_s) = \mathbf{0}$ is a zero equilibrium point of the secondary system (18).

So far, the nonlinear nonminimum phase tracking system (15) is decomposed into two systems by the additive state decomposition, where the external signal $\mathbf{d} + \varphi$ is shifted to (17) and the nonlinear term $\phi(\cdot)$ is shifted to (18). The strategy here is to assign the tracking (rejection) task to the primary system (17) and stabilization task to the secondary system (18), as shown in Fig. 3. More concretely, in (17) design v_p to track r , and design v_s to stabilize (18). If so, by the relationship (19), y can track r . In the following, controllers v_p and v_s are designed separately. There exist many mature methods for nonlinear system state feedback stabilization, such as backstepping technique. It is well known that nonminimum phase behavior restricts the application of basic nonlinear controllers. However, by the proposed decomposition, we only need to consider state feedback stabilization for (18), namely nonminimum phase behavior is avoided for the stabilization problem. Hence, the advantage of the proposed method lies in the decomposition of the problem into two well-studied control problems.

3.3. Tracking controller design for primary system

Before proceeding further, we have the following preliminary result. Consider the following linear system:

$$\begin{aligned} \dot{\mathbf{z}}_1 &= \mathbf{S}_z \mathbf{z}_1 + \mathbf{A}_{12} e_z \\ \dot{\mathbf{z}}_2 &= \mathbf{A}_{21} \mathbf{z}_1 + \mathbf{A}_{22} \mathbf{z}_2 + \mathbf{d}_1 + \varphi_1 \\ e_z &= \mathbf{C}_e^T \mathbf{z}_2 + d_2 + \varphi_2, \quad \mathbf{z}(0) = \mathbf{z}_0 \end{aligned} \quad (20)$$

where $\mathbf{S}_z \in \mathbb{R}^{m_1 \times m_1}$ is a marginally stable matrix, $\mathbf{A}_{12} \in \mathbb{R}^{m_1}$, $\mathbf{C}_e \in \mathbb{R}^{m_2}$, $\mathbf{A}_{21} \in \mathbb{R}^{m_2 \times m_1}$, $\mathbf{A}_{22} \in \mathbb{R}^{m_2 \times m_2}$, $\mathbf{z}_1 \in \mathbb{R}^{m_1}$, $\mathbf{z}_2, \mathbf{d}_1, \varphi_1 \in \mathbb{R}^{m_2}$, $\mathbf{z} = [\mathbf{z}_1^T \mathbf{z}_2^T]^T \in \mathbb{R}^{m_1+m_2}$ and $e_z, d_2, \varphi_2 \in \mathbb{R}$.

Lemma 1. Suppose (i) $\varphi_1(t), \varphi_2(t)$ are bounded on $[0, \infty)$ and $\lim_{t \rightarrow \infty} \|\varphi_1(t)\| = 0$, $\lim_{t \rightarrow \infty} \varphi_2(t) = 0$, (ii) every element of $\mathbf{d}_1(t), d_2(t)$ is bounded on $[0, \infty)$ and can be generated by $\dot{\mathbf{w}}_z = \mathbf{S}_z \mathbf{w}_z$, $d_z = \mathbf{C}_z^T \mathbf{w}_z$ with appropriate initial values, where

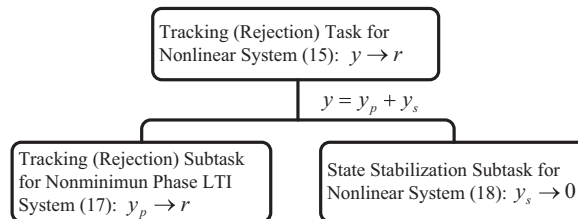


Fig. 3. Additive state decomposition of the TORA system.

$\mathbf{C}_z, \mathbf{w}_z \in \mathbb{R}^{m_1}$, (iii) the parameters in (20) satisfy

$$\max \operatorname{Re} \lambda(\mathbf{A}_z) < 0, \quad \mathbf{A}_z = \begin{bmatrix} \mathbf{S}_z & \mathbf{A}_{12} \mathbf{C}_e^T \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}. \quad (21)$$

Then in (20) $\lim_{t \rightarrow \infty} e_z(t) = 0$, meanwhile keeping $\mathbf{z}_1(t)$ and $\mathbf{z}_2(t)$ bounded.

Proof. See Appendix B.

Define a filtered tracking error to be

$$e_p = \hat{y}_p + a \dot{\hat{y}}_p = (\mathbf{C} + a\mathbf{B})^T \mathbf{x}_p - r \quad (22)$$

where $\hat{y}_p = y_p - r$, $\dot{r} = 0$ and $a > 0$. Let us consider the tracking problem for the primary system (17). With Lemma 1 in hand, the design of v_p is stated in Theorem 2.

Theorem 2. For the primary system (17), let the controller v_p be designed as follows:

$$\begin{aligned} \dot{\xi} &= \mathbf{S}_a \xi + \mathbf{L}_1 e_p \\ v_p(\xi, \mathbf{x}_p, r) &= \mathbf{L}_2^T \mathbf{x}_p + \mathbf{L}_3^T \xi \end{aligned} \quad (23)$$

where $\mathbf{S}_a = \operatorname{diag}(0, \mathbf{S})$, $\mathbf{L}_1 \in \mathbb{R}^{m+1}$, $\mathbf{L}_2 \in \mathbb{R}^4$ and $\mathbf{L}_3 \in \mathbb{R}^{m+1}$ satisfy

$$\max \operatorname{Re} \lambda(\mathbf{A}_a) < 0, \quad \mathbf{A}_a = \begin{bmatrix} \mathbf{S}_a & \mathbf{L}_1(\mathbf{C} + a\mathbf{B})^T \\ \mathbf{B}\mathbf{L}_3^T & \mathbf{A} + \mathbf{B}\mathbf{L}_2^T \end{bmatrix}. \quad (24)$$

Then $\lim_{t \rightarrow \infty} y_p(t) = r$ and $\lim_{t \rightarrow \infty} \dot{y}_p(t) = 0$ meanwhile keeping $\mathbf{x}_p(t)$ and $\xi(t)$ bounded.

Proof. Incorporating the controller (23) into the primary system (17) results in

$$\begin{aligned} \dot{\xi} &= \mathbf{S}_a \xi + \mathbf{L}_1 e_p \\ \dot{\mathbf{x}}_p &= (\mathbf{A} + \mathbf{B}\mathbf{L}_2^T) \mathbf{x}_p + \mathbf{B}\mathbf{L}_3^T \xi + \mathbf{d} + \varphi \\ e_p &= (\mathbf{C} + a\mathbf{B})^T \mathbf{x}_p - r \end{aligned}$$

where the definition (22) is utilized. Moreover, every element of $\mathbf{d}$ and r can be generated by an autonomous system in the form $\dot{\mathbf{w}}_a = \mathbf{S}_a \mathbf{w}_a$, $d_a = \mathbf{C}_a^T \mathbf{w}_a$ with appropriate initial values, where $\mathbf{C}_a = [1 \ \mathbf{C}_d^T]^T$. By Lemma 1, if (24) holds, then $\lim_{t \rightarrow \infty} e_p(t) = 0$ meanwhile keeping $\mathbf{x}_p(t)$ and $\xi(t)$ bounded. It is easy to see from (22) that both $\hat{y}_p$ and $\dot{\hat{y}}_p$ can be viewed as outputs of a stable system with e_p as input. This means that $\hat{y}_p$ and $\dot{\hat{y}}_p$ are bounded if e_p is bounded. In addition, $\lim_{t \rightarrow \infty} \hat{y}_p(t) = 0$ and $\lim_{t \rightarrow \infty} \dot{\hat{y}}_p(t) = 0$. $\square$

In most of cases, the controller parameters $\mathbf{L}_1, \mathbf{L}_2$ and $\mathbf{L}_3$ in (23) can always be found. This is shown in the following proposition.

Proposition 1. For any $\mathbf{S} = -\mathbf{S}^T$ without eigenvalues $\pm j$, the parameters

$$\mathbf{L}_1 = [1 \ \mathbf{C}_d^T]^T, \quad \mathbf{L}_2 = -\frac{1}{a} \mathbf{C} - \mathbf{B} - \frac{1}{a} \mathbf{H} - \frac{1}{a} \mathbf{K}, \quad \mathbf{L}_3 = -\frac{1}{a} \mathbf{L}_1 \quad (25)$$

can always make $\max \operatorname{Re} \lambda(\mathbf{A}_a) < 0$, where $\mathbf{H} = [0 \ \varepsilon \ 0 \ 1]^T$.

Proof. See Appendix C.

Remark 5. Proposition 1 in fact implies that, in the presence of any external signal (except for the frequencies at ± 1), the controller (23) with parameters (25) can always make $\lim_{t \rightarrow \infty} y_p(t) = r$ and $\lim_{t \rightarrow \infty} \dot{y}_p(t) = 0$ meanwhile keeping $\mathbf{x}_p(t)$ and $\xi(t)$ bounded. In other words, the disturbance like $\sin t$ cannot be dealt with, which is consistent with [9]. If the external signal contains the component with frequencies at ± 1 , then such a frequency component can be chosen not to compensate for, i.e., $\mathbf{S}_a$ in (23) will not contain eigenvalues $\pm j$.

3.4. Stabilized controller design for secondary system

So far, we have designed the tracking controller for the primary system (17). In this section, we are going to design the stabilized controller for the secondary system (18). It can be rewritten as

$$\begin{aligned} \dot{x}_{1,s} &= x_{2,s} \\ \dot{x}_{2,s} &= -x_{1,s} + \varepsilon \sin(x_{3,s} + r) - \varepsilon \sin r + g \\ \dot{x}_{3,s} &= x_{4,s} \\ \dot{x}_{4,s} &= \mathbf{K}^T \mathbf{x}_s + v_s, \quad \mathbf{x}_s(0) = 0 \end{aligned} \quad (26)$$

where $g = \varepsilon \sin(y_p + x_{3,s}) - \varepsilon \sin(r + x_{3,s}) - \varepsilon(y_p + a\dot{y}_p - r)$. Our constructive procedure has been inspired by the design in [4]. We will start the controller design procedure from the marginally stable $(x_{1,s}, x_{2,s})$ -subsystem.

Step 1: Consider the $(x_{1,s}, x_{2,s})$ -subsystem of (26) with $x_{3,s}$ viewed as the virtual control input. Differentiating the quadratic function $V_1 = \frac{1}{2}(x_{1,s}^2 + x_{2,s}^2)$ results in

$$\dot{V}_1 = \varepsilon x_{2,s} [\sin(x_{3,s} + r) - \sin r] + \varepsilon x_{2,s} g. \quad (27)$$

Guided by the state-feedback design [3], we introduce the following “Certainty Equivalence” (CE) based virtual controller

$$x_{3,s} = -b \operatorname{atan} x_{2,s} + x'_{3,s}. \quad (28)$$

Then

$$\begin{aligned} \dot{x}_{1,s} &= x_{2,s} \\ \dot{x}_{2,s} &= -x_{1,s} + 2\varepsilon \sin\left(\frac{-b \operatorname{atan} x_{2,s}}{2}\right) \cos\left(\frac{-b \operatorname{atan} x_{2,s} + 2r}{2}\right) + g' \end{aligned} \quad (29)$$

where

$$g' = \varepsilon \sin(r - b \operatorname{atan} x_{2,s} + x'_{3,s}) - \varepsilon \sin(r - b \operatorname{atan} x_{2,s}) + g. \quad (30)$$

In order to ensure $\cos((-b \operatorname{atan} x_{2,s} + 2r)/2) > 0$, the parameter b is chosen to satisfy $0 < b < 2(1 - |r|/\pi)$. Since $r \in (-\pi/2, \pi/2)$ is a constant, b always exists. The term CE is used here because $x'_{3,s} = 0$ in (28) makes $\dot{V}_1$ in (27) negative semidefinite as $g \equiv 0$.

Step 2: We will apply backstepping to the $(x'_{3,s}, x_{4,s})$ -subsystem and design a nonlinear controller v_s to drive $x'_{3,s}$ to the origin. By the definition (28), $x'_{3,s} = x_{3,s} + b \operatorname{atan} x_{2,s}$. Then the time derivative of the new variable $x'_{3,s}$ is

$$\dot{x}'_{3,s} = x_{4,s} + \psi + b \frac{1}{1 + x_{2,s}^2} g \quad (31)$$

where $\psi = b(1/(1 + x_{2,s}^2))[-x_{1,s} + \varepsilon \sin(x_{3,s} + r) - \varepsilon \sin r]$. Define a new variable $x'_{4,s}$ as follows:

$$x'_{4,s} = x'_{3,s} + x_{4,s} + \psi. \quad (32)$$

Then (31) becomes

$$\dot{x}'_{3,s} = -x'_{3,s} + x'_{4,s} + b \frac{1}{1 + x_{2,s}^2} g.$$

By the definition (32), the time derivative of the new variable $x'_{4,s}$ is

$$\begin{aligned} \dot{x}'_{4,s} &= \dot{x}'_{3,s} + \dot{x}_{4,s} + \dot{\psi} \\ &= -x'_{3,s} + x'_{4,s} + b \frac{1}{1 + x_{2,s}^2} g + \mathbf{K}^T \mathbf{x}_s + v_s + \dot{\psi}, \end{aligned}$$

where

$$\dot{\psi} = -2bx_{2,s} \frac{1}{(1 + x_{2,s}^2)^2} [-x_{1,s} + \varepsilon \sin(x_{3,s} + r) - \varepsilon \sin r] + b \frac{1}{1 + x_{2,s}^2} [-x_{2,s} + \varepsilon \cos(x_{3,s} + r)x_{4,s}].$$

Design v_s for the secondary system (26) as follows:

$$v_s(\mathbf{x}_p, \mathbf{x}_s, r) = x'_{3,s} - 2x'_{4,s} - \mathbf{K}^T \mathbf{x}_s - \dot{\psi}. \quad (33)$$

Then the $(x'_{3,s}, x'_{4,s})$ -subsystem becomes

$$\begin{aligned} \dot{x}'_{3,s} &= -x'_{3,s} + x'_{4,s} + b \frac{1}{1 + x_{2,s}^2} g \\ \dot{x}'_{4,s} &= -x'_{4,s} + b \frac{1}{1 + x_{2,s}^2} g. \end{aligned} \quad (34)$$

It is easy to see that $\lim_{t \rightarrow \infty} x'_{3,s}(t) = 0$ and $\lim_{t \rightarrow \infty} x'_{4,s}(t) = 0$ as $\lim_{t \rightarrow \infty} \operatorname{lim} g(t) = 0$.

We are now ready to state the theorem for the secondary system.

Theorem 3. Suppose $\lim_{t \rightarrow \infty} y_p(t) = r$ and $\lim_{t \rightarrow \infty} \dot{y}_p(t) = 0$. Let the controller v_s for the secondary system (26) be designed as (33), where $0 < b < 2(1 - 2||r||/\pi)$. Then $\lim_{t \rightarrow \infty} ||\mathbf{x}_s(t)|| = 0$ meanwhile keeping $\mathbf{x}_s(t)$ bounded.

Proof. See Appendix D.

3.5. Controller synthesis for original system

It should be noticed that the controller design above is based on the condition that $\mathbf{x}_p$ and $\mathbf{x}_s$ are known a priori. A problem arises that the states $\mathbf{x}_p$ and $\mathbf{x}_s$ cannot be measured directly except for $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_s$. By taking this into account, the following observer is proposed to estimate the states $\mathbf{x}_p$ and $\mathbf{x}_s$, which is stated in Theorem 4.

Theorem 4. Let the observer be designed as follows:

$$\begin{aligned}\dot{\hat{\mathbf{x}}}_s &= \mathbf{A}\hat{\mathbf{x}}_s + \mathbf{B}v_s + \phi(y, \dot{y}) - \phi(r, 0) \\ \hat{\mathbf{x}}_p &= \mathbf{x} - \hat{\mathbf{x}}_s, \quad \hat{\mathbf{x}}_s(0) = \mathbf{0}\end{aligned}\quad (35)$$

where $\mathbf{A}$ is stable. Then $\hat{\mathbf{x}}_p \equiv \mathbf{x}_p$ and $\hat{\mathbf{x}}_s \equiv \mathbf{x}_s$.

Proof. Since $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_s$, we have $y = y_p + y_s$. Consequently, (35) can be rewritten as

$$\begin{aligned}\dot{\hat{\mathbf{x}}}_s &= \mathbf{A}\hat{\mathbf{x}}_s + \mathbf{B}v_s + \phi(y_p + y_s, \dot{y}_p + \dot{y}_s) - \phi(r, 0) \\ \hat{\mathbf{x}}_p &= \mathbf{x} - \hat{\mathbf{x}}_s, \quad \hat{\mathbf{x}}_s(0) = \mathbf{0}.\end{aligned}\quad (36)$$

Subtracting (18) from (36) results in

$$\dot{\tilde{\mathbf{x}}}_s = \mathbf{A}\tilde{\mathbf{x}}_s, \quad \tilde{\mathbf{x}}_s(0) = \mathbf{0}\quad (37)$$

where $\tilde{\mathbf{x}}_s = \hat{\mathbf{x}}_s - \mathbf{x}_s$. Then $\hat{\mathbf{x}}_s \equiv \mathbf{x}_s$. Furthermore, with the aid of the relationship $\mathbf{x}_p = \mathbf{x} - \mathbf{x}_s$, we have $\hat{\mathbf{x}}_p \equiv \mathbf{x}_p$. $\square$

Remark 6. Unlike traditional observers, the proposed observer can estimate the states of the primary system and the secondary system directly rather than asymptotically or exponentially. This can be explained that, although the initial value $\mathbf{x}_0$ is unknown, the initial value of either the primary system or the secondary system can be specified exactly, leaving an unknown initial value for the other system. The measurement $\mathbf{x}$ and parameters may be inaccurate. In this case, it is expected that small uncertainties lead to $\hat{\mathbf{x}}_p$ close to $\mathbf{x}_p$ (or $\hat{\mathbf{x}}_s$ close to $\mathbf{x}_s$). From (37), a stable matrix $\mathbf{A}$ can ensure a small $\tilde{\mathbf{x}}_s$ in the presence of small uncertainties.

Theorem 5. Suppose that the conditions of Theorems 1–4 hold. Let the controller u in the system (1) be designed as follows:

$$\begin{aligned}\dot{\xi} &= \mathbf{S}_d \xi + \mathbf{L}_1[(\mathbf{C} + a\mathbf{B})^T \hat{\mathbf{x}}_p - r] \\ u &= \mathbf{K}^T \mathbf{x} + v_p(\xi, \hat{\mathbf{x}}_p, r) + v_s(\hat{\mathbf{x}}_p, \hat{\mathbf{x}}_s, r) + \frac{\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} \hat{F}_d\end{aligned}\quad (38)$$

meanwhile keeping $\mathbf{x}$ and ξ bounded.

Proof. Note that the original system (1), the primary system (17) and the secondary system (18) have the relationship: $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_s$ and $y = y_p + y_s$. With the controller (38), for the primary system (17), $\lim_{t \rightarrow \infty} y_p(t) = r$ meanwhile keeping $\mathbf{x}_p$ and ξ bounded by Theorem 2. On the other hand, for the secondary system (18), we have $\lim_{t \rightarrow \infty} \|\mathbf{x}_s(t)\| = 0$ meanwhile keeping $\mathbf{x}_s$ bounded on $[0, \infty)$ by Theorem 3. In addition, Theorem 4 ensures that $\hat{\mathbf{x}}_p \equiv \mathbf{x}_p$ and $\hat{\mathbf{x}}_s \equiv \mathbf{x}_s$. Therefore, $\lim_{t \rightarrow \infty} y(t) = r$ meanwhile keeping $\mathbf{x}$ and ξ bounded. $\square$

4. Numerical simulation

In the simulation, set $\varepsilon = 0.2$ and the initial value $\mathbf{x}_0 = [0 \ 0 \ 0 \ 0]^T$ in (1). The unknown dimensionless disturbance F_d is generated by an autonomous LTI system (2) with the parameters as follows:

$$\mathbf{S} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad \mathbf{C}_d = [1 \ 0]^T, \quad \mathbf{w}(0) = [0 \ 0.02]^T.$$

The objective here is to design a controller u such that the output $y(t) = x_3(t) \rightarrow r = 0.5$ as $t \rightarrow \infty$ meanwhile keeping the other states bounded.

The parameters of the observer (13) are chosen as $l_1 = l_2 = 10$. In (16), the parameters of $\mathbf{A}$ are chosen as $a = 1$ and $\mathbf{K} = [0 \ -\varepsilon \ -1 \ -2]^T$. Then $\max \operatorname{Re}(\lambda(\mathbf{A})) = -0.01 < 0$. Since matrix $\mathbf{S}$ does not possess the eigenvalues $\pm j$, the parameters of the tracking controller (23) of the primary system can be chosen according to Proposition 1 that $\mathbf{L}_1 = [1 \ \mathbf{C}_d^T]^T$, $\mathbf{L}_2 = \mathbf{0}$ and $\mathbf{L}_3 = -\mathbf{L}_1$. Then $\max \operatorname{Re}(\lambda(\mathbf{A}_d)) = -0.0084 < 0$. The parameter b of the stabilized controller (33) is chosen as $b = 1.5(1 - 1/\pi) < 2(1 - 2|r|/\pi)$.

The TORA system (1) is driven by the controller (38) with the parameters above. The evolutions of all states of (1) are shown in Fig. 4. As shown, the proposed controller u drives the output $y(t) = x_3(t) \rightarrow 0.5$ as $t \rightarrow \infty$, meanwhile keeping the other states bounded. Moreover, the control signal is smooth and bounded.

Unlike the output regulation theory, the proposed method does not require the regulator equations. If the disturbance F_d consists of more frequency components, i.e., $\mathbf{S}$ is more complicated, the designed controller above does not need to be changed except for the corresponding $\mathbf{S}$ and $\mathbf{C}_d$. This demonstrates the effectiveness of the proposed control method.

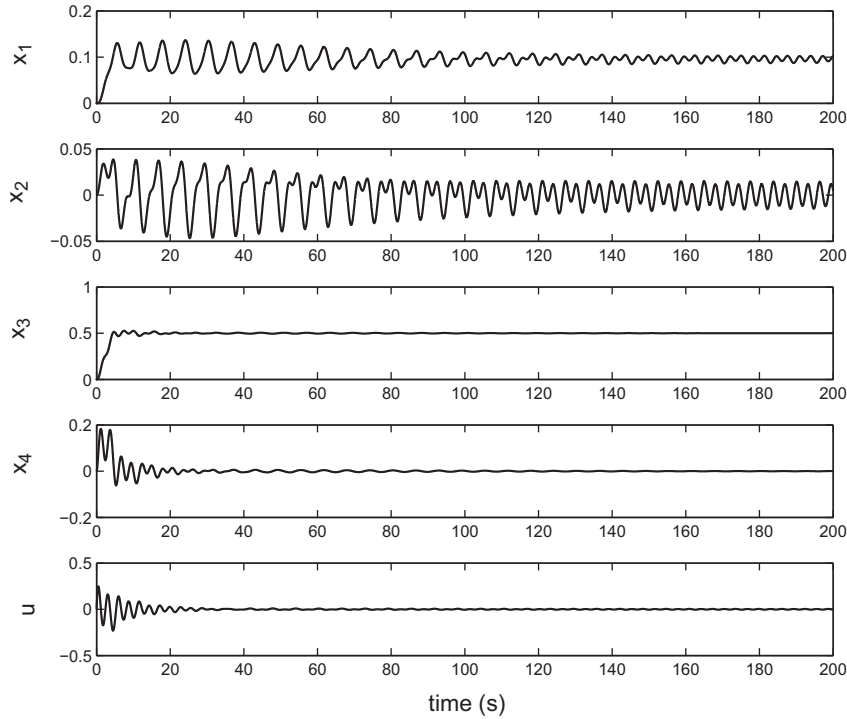


Fig. 4. Evolutions of all states.

For example, we consider that the unknown dimensionless disturbance F_d is generated by an autonomous LTI system (2) with the parameters as follows:

$$\mathbf{S} = \text{diag}(\mathbf{S}_1, \mathbf{S}_2), \quad \mathbf{S}_1 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix} 0 & 1.5 \\ -1.5 & 0 \end{bmatrix}, \quad \mathbf{C}_d = [1 \ 0 \ 1 \ 0]^T, \quad \mathbf{w}(0) = [0 \ 0.02 \ 0 \ 0.02]^T.$$

The controller in the first simulation is still applied to this case except for replacing $\mathbf{S}$ and $\mathbf{C}_d$ (the dimension is changed correspondingly). Driven by the new controller, the evolutions of all states of (1) are shown in Fig. 5. As shown, the proposed controller u drives the output $y(t) = x_3(t) \rightarrow 0.5$ as $t \rightarrow \infty$, meanwhile keeping the other states bounded. Moreover, the control signal is smooth and bounded.

5. Discussions

The idea of the proposed additive state decomposition has been implicitly mentioned in existing literature. For example, a commonly used step to transform a tracking problem to a stabilization problem has implicitly used the additive state decomposition, where the reference system presents the primary system, leaving the error dynamics to be the secondary system. The decomposition here is with respect to state. So, we call it “additive state decomposition”. The authors also proposed “additive output decomposition” [16], namely the decomposition is with respect to output. By additive state decomposition, the output tracking task is decomposed into an output tracking subtask for an LTI system and a state feedback stabilization subtask for a nonlinear system. Then one can design a controller for each subtask, respectively; these are combined together to achieve the original control task. The tensor product (TP) model transformation [5,17,18] can offer us another way to decompose the original system, namely two decomposed subsystems are both nonlinear but with one in the form of TP model. The TP model can be considered as a class of linear-parameter-varying models by the convex combination of linear LTI models. According to this, it is easier to deal with than a general nonlinear system. The additive-state-decomposition-based control design methodology is in fact a framework rather than a control technology. By such a framework, a tracking problem is decomposed into two well solved problems. Control methods in both the time domain and frequency domain are applicable to the component systems. Owing to the decomposition, the resulting controller can avoid conflict among tracking performance, rejection performance and robustness. This is similar to the idea of two-degree-of-freedom control [19]. For further information on additive-state-decomposition-based control, readers can refer to [20–22].

The TORA is often of independent interest to serve as a benchmark for different nonlinear control methods. Some of researchers have focused on global stabilization [18,23–25]. These methods can be applied to the stabilized controller design for the secondary system under the framework of the proposed additive-state-decomposition-based control design. Due to the decomposition, we can verify the performance of the designed stabilized controller separately. The difficulties will

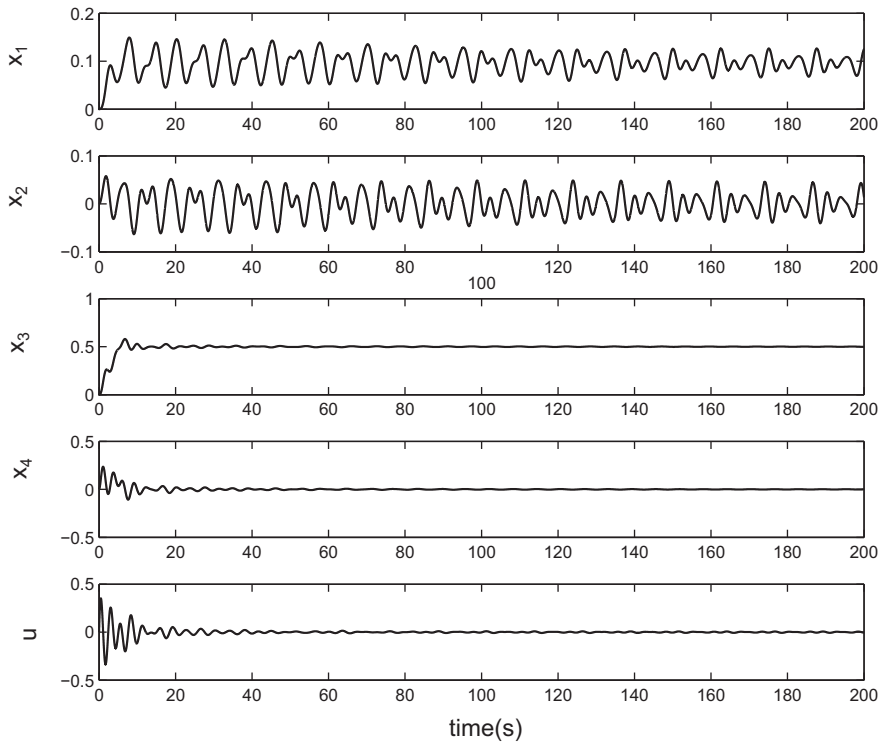


Fig. 5. Evolutions of all states subject to a complicated disturbance.

obviously increase as more general control problems such as output tracking and disturbance rejection are considered. Motivated by this, the proposed additive-state-decomposition-based control aims to reduce the difficulties. In this paper, we give the solutions to the controller design for the primary system and secondary system, but in fact do not have to limit them. With different solutions, the resulting controller may achieve better performance and different requirements.

6. Conclusions

In this paper, the tracking (rejection) problem for rotational position of the TORA was discussed. Our main contribution lies in the presentation of a new decomposition scheme, named additive state decomposition, which not only simplifies the controller design but also increases flexibility of the controller design. By the additive state decomposition, the considered system was decomposed into two subsystems in charge of two independent subtasks, respectively: an LTI system in charge of a tracking (rejection) subtask, leaving a nonlinear system in charge of a stabilization subtask. Based on the decomposition, the subcontrollers corresponding to two subsystems were designed separately, which increased the flexibility of design. The tracking (rejection) controller was designed by the transfer function method, while the stabilized controller was designed by the backstepping method. This numerical simulation has shown that the designed controller can achieve the objective, moreover, can be changed flexibly according to the model of external signals.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (61104012), the 973 Program (2010CB327904), and Specialized Research Fund for the Doctoral Program of Higher Education (20111102120008).

Appendix A. Proof of Theorem 1

The disturbance $F_d \in \mathbb{R}$ is generated by an autonomous LTI system (2) with an initial value $\mathbf{w}(0)$. It can also be generated by the following system:

$$\dot{\mathbf{w}} = \mathbf{S}\mathbf{w}, \quad F_d = l_1 \mathbf{C}_d^T \mathbf{w} \quad (\text{A.1})$$

with the initial value $(1/l_1)\mathbf{w}(0)$. Subtracting ((1d) and A.1) from (13) results in

$$\begin{aligned}\dot{\tilde{\mathbf{w}}} &= \mathbf{S}\tilde{\mathbf{w}} + l_1 \frac{\varepsilon \cos x_3}{1-\varepsilon^2 \cos^2 x_3} \mathbf{C}_d \tilde{x}_4 \\ \dot{\tilde{x}}_4 &= -l_2 \tilde{x}_4 - l_1 \frac{\varepsilon \cos x_3}{1-\varepsilon^2 \cos^2 x_3} \mathbf{C}_d^T \tilde{\mathbf{w}}\end{aligned}\quad (\text{A.2})$$

where $l_1, l_2 > 0$, $\tilde{x}_4 \triangleq \hat{x}_4 - x_4$ and $\tilde{\mathbf{w}} \triangleq \hat{\mathbf{w}} - \mathbf{w}$. Design a Lyapunov function as follows:

$$V_1 = \frac{1}{2} \tilde{\mathbf{w}}^T \tilde{\mathbf{w}} + \frac{1}{2} \tilde{x}_4^2.$$

Taking the derivative of V_1 along (A.2) results in

$$\begin{aligned}\dot{V}_1 &= \frac{1}{2} \tilde{\mathbf{w}}^T (\mathbf{S} + \mathbf{S}^T) \tilde{\mathbf{w}} + l_1 \tilde{\mathbf{w}}^T \frac{\varepsilon \cos x_3}{1-\varepsilon^2 \cos^2 x_3} \mathbf{C}_d \tilde{x}_4 \\ &\quad - l_2 \tilde{x}_4^2 - l_1 \tilde{x}_4 \frac{\varepsilon \cos x_3}{1-\varepsilon^2 \cos^2 x_3} \mathbf{C}_d^T \tilde{\mathbf{w}}.\end{aligned}$$

By Assumption 2, $\mathbf{S} + \mathbf{S}^T = \mathbf{0}$. Then the derivative of V_1 becomes

$$\dot{V}_1 \leq -l_2 \tilde{x}_4^2 \leq 0.$$

Since $l_2 > 0$, from the inequality above, it can be concluded by LaSalle's invariance principle [14] that $\lim_{t \rightarrow \infty} \tilde{x}_4(t) = 0$ and $\lim_{t \rightarrow \infty} (\varepsilon \cos x_3 / (1 - \varepsilon^2 \cos^2 x_3)) \mathbf{C}_d^T \tilde{\mathbf{w}}(t) = 0$.

Appendix B. Proof of Lemma 1

Before proving Lemma 1, we need the following preliminary result.

Lemma 2. If the pair $(\mathbf{A}_z, \mathbf{B}_z)$ is controllable, then there exists a $\mathbf{C}_0 \in \mathbb{R}^m$ such that

$$\mathbf{C}_0^T (\mathbf{sI}_m - \mathbf{A}_z)^{-1} \mathbf{B}_z = \frac{1}{\det(\mathbf{sI}_m - \mathbf{A}_z)}$$

where $\mathbf{A}_z \in \mathbb{R}^{m \times m}$ and $\mathbf{B}_z \in \mathbb{R}^m$.

Proof. First, we have

$$(\mathbf{sI}_m - \mathbf{A}_z)^{-1} \mathbf{B}_z = \mathbf{N} [\mathbf{s}^{n-1} \ \dots \ 1]^T / \det(\mathbf{sI}_m - \mathbf{A}_z)$$

where $\mathbf{N} \in \mathbb{R}^{m \times m}$. If the pair $(\mathbf{A}_z, \mathbf{B}_z)$ is controllable, the matrix $\mathbf{N}$ is full rank [13]. We can complete this proof by choosing $\mathbf{C}_0 = (\mathbf{N}^{-1})^T [0 \ \dots \ 0 \ 1]^T$. $\square$

With Lemma 2 in hand, we are ready to prove Lemma 1.

(i) For the system (20), we have

$$\mathbf{z}(t) = e^{-\mathbf{A}_z t} \mathbf{z}_0 + \int_0^t e^{-\mathbf{A}_z(t-\tau)} (\mathbf{d}_a + \varphi_a)(\tau) \, d\tau, \quad t \geq 0$$

where $\mathbf{d}_a = [\mathbf{d}_2^T \mathbf{A}_{12}^T \ \mathbf{d}_1^T]^T$ and $\varphi_a = [\varphi_2^T \mathbf{A}_{12}^T \ \varphi_1^T]^T$. Based on the equation above, since $\lambda(\mathbf{A}_z) < 0$ and $\|\mathbf{d}_a(t)\|, \|\varphi_a(t)\|$ are bounded on $[0, \infty)$, it is easy to see that $\|\mathbf{z}_1(t)\|$ and $\|\mathbf{z}_2(t)\|$ are bounded on $[0, \infty)$.

(ii) For the system (20), the Laplace transformation of $\mathbf{z}(s)$ is

$$\mathbf{z}(s) = (\mathbf{sI}_{m_1+m_2} - \mathbf{A}_z)^{-1} [\mathbf{d}_a(s) + \varphi_a(s) + \mathbf{z}_0].$$

Then $\mathbf{z}_1(s) = \mathbf{C}_a^T (\mathbf{sI}_{m_1+m_2} - \mathbf{A}_z)^{-1} [\mathbf{d}_a(s) + \varphi_a(s) + \mathbf{z}_0]$, where $\mathbf{C}_a = [\mathbf{I}_{m_1} \ \mathbf{0}_{m_1 \times m_2}]^T$. The condition $\lambda(\mathbf{A}_z) < 0$ implies that the pair $(\mathbf{S}_z, \mathbf{A}_{12})$ is controllable. Otherwise, for the autonomous system $\dot{\mathbf{z}} = \mathbf{A}_z \mathbf{z}$, the variable $\mathbf{z}_1$ cannot converge to zero as $\mathbf{S}_z$ is a marginally stable matrix. This contradicts with the condition $\lambda(\mathbf{A}_z) < 0$. Then by Lemma 1, there exists a $\mathbf{C}_0 \in \mathbb{R}^{m_1}$ such that

$$\mathbf{C}_0^T \mathbf{z}_1(s) = \frac{1}{\det(\mathbf{sI}_{m_1} - \mathbf{S}_z)} e_z(s).$$

Then $e_z(s)$ can be written as

$$\begin{aligned}e_z(s) &= \det(\mathbf{sI}_{m_1} - \mathbf{S}_z) \mathbf{C}_0^T \mathbf{z}_1(s) \\ &= \mathbf{Q}(s) \det(\mathbf{sI}_{m_1} - \mathbf{S}_z) [\mathbf{d}_a(s) + \varphi_a(s) + \mathbf{z}_0]\end{aligned}$$

where $\mathbf{Q}(s) = \mathbf{C}_0^T \mathbf{C}_a^{-1} (\mathbf{sI}_{m_1+m_2} - \mathbf{A}_z)^{-1} \mathbf{D}_a$. Since every element of $\mathbf{d}_a$ can be generated by $\dot{\mathbf{w}}_z = \mathbf{S}_z \mathbf{w}_z, d_z = \mathbf{C}_z^T \mathbf{w}_z$, we have $\mathbf{d}_a(s) = [\mathbf{C}_z^T (\mathbf{sI}_{m_1} - \mathbf{S}_z)^{-1} \mathbf{w}_{z,i}(0)]_{(m_1+m_2) \times 1}$, where $\mathbf{w}_{z,i}(0) \in \mathbb{R}$. Since $(\mathbf{sI}_{m_1} - \mathbf{S}_z)^{-1} = (1/\det(\mathbf{sI}_{m_1} - \mathbf{S}_z)) \text{adj}(\mathbf{sI}_{m_1} - \mathbf{S}_z)$, $e_z(s)$ is further represented as

$$\mathbf{e}_z(s) = \mathbf{Q}(s) \mathbf{C}_d^T \mathbf{w}_z(0) [\mathbf{C}_z^T \text{adj}(\mathbf{sI}_{m_1} - \mathbf{S}_z) \mathbf{w}_{z,i}(0)]_{(m_1+m_2) \times 1}$$

$$+(\mathbf{S}\mathbf{I}_{m_1+m_2}-\mathbf{A}_z)^{-1}[\boldsymbol{\varphi}_a(s)+\mathbf{z}_0]. \quad (\text{B.1})$$

Since $\lambda(\mathbf{A}_z) < 0$ and the order of $\mathbf{A}_z$ is higher than that of $\mathbf{S}_z$, moreover $\|\boldsymbol{\varphi}_a(t)\|$ is bounded on $[0, \infty)$ and $\lim_{t \rightarrow \infty} \|\boldsymbol{\varphi}_a(t)\| = 0$, for any initial value $w_{z,i}(0)$, we have $\lim_{t \rightarrow \infty} \text{lime}_z(t) = 0$ from (B.1).

Appendix C. Proof of Proposition 1

If we can prove that the following system:

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \mathbf{S}_a \boldsymbol{\xi} + \mathbf{L}_1(\mathbf{C} + a\mathbf{B})^T \mathbf{x}_p \\ \dot{\mathbf{x}}_p &= (\mathbf{A} + \mathbf{B}\mathbf{L}_2^T) \mathbf{x}_p + \mathbf{B}\mathbf{L}_3^T \boldsymbol{\xi} \end{aligned} \quad (\text{C.1})$$

is asymptotic stable, then $\text{Re } \lambda(\mathbf{A}_a) < 0$ holds. Choose a Lyapunov function as follows:

$$V = \frac{1}{2} \boldsymbol{\xi}^T \boldsymbol{\xi} + \frac{1}{2} x_{1,p}^2 + \frac{1}{2} x_{2,p}^2 + \frac{1}{2} p^2$$

where $p = x_{3,p} + ax_{4,p} = (\mathbf{C} + a\mathbf{B})^T \mathbf{x}_p$. With the parameters $\mathbf{L}_1 = [\mathbf{C}_d^T]^T$, $\mathbf{L}_2 = -(1/a)\mathbf{C} - \mathbf{B} - (1/a)\mathbf{H} - \mathbf{K}$ and $\mathbf{L}_3 = -(1/a)\mathbf{L}_1$, the derivative of V along (C.1) is

$$\dot{V} = -p^2.$$

Define $\mathcal{S} = \{\mathbf{x} | \dot{V}(\mathbf{x}) = 0\}$, where $\mathbf{x} = [\boldsymbol{\xi}^T \mathbf{x}_p^T]^T$. The remaindering work is to prove $\mathcal{S} = \{\mathbf{x} | \mathbf{x} = 0\}$. If so, by LaSalle's invariance principle [14], we have $\lim_{t \rightarrow \infty} \|\mathbf{x}(t)\| = 0$. Therefore, the system (C.1) with the parameters is globally asymptotically stable. Then $\text{Re } \lambda(\mathbf{A}_a) < 0$.

Since $\dot{V} = 0 \Rightarrow p = 0$ and $a > 0$, we have $\mathcal{S} = \{\mathbf{x} | x_{3,p} = x_{4,p} = 0\}$. Let $\mathbf{x}$ be a solution belonging to $\mathcal{S}$ identically. Then, from (C.1), we have

$$\dot{\boldsymbol{\xi}} = \mathbf{S}_a \boldsymbol{\xi} \quad (\text{C.2})$$

$$\dot{x}_{1,p} = x_{2,p}, \quad \dot{x}_{2,p} = -x_{1,p} \quad (\text{C.3})$$

$$0 = -\epsilon x_{2,p} + \mathbf{L}_3^T \boldsymbol{\xi} \quad (\text{C.4})$$

From (C.3), it holds that

$$x_{2,p} \in \mathcal{S}_1 = \left\{ [0 \ 1] \boldsymbol{\xi} \mid \boldsymbol{\xi} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boldsymbol{\xi} \right\}.$$

On the other hand, from (C.2) and (C.4), it holds that

$$x_{2,p} \in \mathcal{S}_2 = \left\{ \frac{1}{\epsilon} \mathbf{L}_3^T \boldsymbol{\xi} \mid \dot{\boldsymbol{\xi}} = \mathbf{S}_a \boldsymbol{\xi} \right\}$$

where matrix $\mathbf{S}_a$ does not possess eigenvalues $\pm j$ since matrix $\mathbf{S}$ does not. Therefore $x_{2,p} \in \mathcal{S}_1 \cap \mathcal{S}_2 = \{0\}$ and then $x_{1,p} = 0$, namely $\mathcal{S} = \{\mathbf{x} | \mathbf{x}_p = 0\}$.

Let $\mathbf{x}$ be a solution that belongs identically to $\mathcal{S}$. Then $\mathbf{L}_3^T \boldsymbol{\xi} = 0$ by (C.4). Since the pair $(\mathbf{C}_d^T, \mathbf{S})$ is observable, by the definition $\mathbf{L}_1 = -(1/a)[\mathbf{C}_d^T]^T$, the pair $(\mathbf{L}_3^T, \mathbf{S}_a)$ is observable as well. Consequently, we can conclude that $\boldsymbol{\xi} = 0$, namely $\mathcal{S} = \{\mathbf{x} | \mathbf{x} = [\boldsymbol{\xi}^T \mathbf{x}_p^T]^T = 0\}$.

Appendix D. Proof of Theorem 3

This proof is composed of three parts.

Part 1. $\lim_{t \rightarrow \infty} x'_{3,s}(t) = 0$, $\lim_{t \rightarrow \infty} x'_{4,s}(t) = 0$ and $\lim_{t \rightarrow \infty} g'(t) = 0$ as $\lim_{t \rightarrow \infty} g(t) = 0$. If $\lim_{t \rightarrow \infty} y_p(t) = r$ and $\lim_{t \rightarrow \infty} \dot{y}_p(t) = 0$, then from the definition of $\phi(y, \dot{y})$, we have $\lim_{t \rightarrow \infty} g(t) = 0$ no matter what y_s is. According to this, it is easy from (34) to see that $\lim_{t \rightarrow \infty} x'_{3,s}(t) = 0$ and $\lim_{t \rightarrow \infty} x'_{4,s}(t) = 0$ when the controller v_s for the secondary system (26) is designed as (33). Then, in (29), $\lim_{t \rightarrow \infty} g'(t) = 0$.

Part 2. $\lim_{t \rightarrow \infty} x_{1,s}(t) = 0$ and $\lim_{t \rightarrow \infty} x_{2,s}(t) = 0$. Since $0 < b < 2(1 - 2\|r\|/\pi)$, the derivative V_1 in (27) negative semidefinite when $g'(t) \equiv 0$, namely,

$$\dot{V}_1 = 2\epsilon x_{2,s} \sin\left(\frac{-b \text{atan } x_{2,s}}{2}\right) \cos\left(\frac{-b \text{atan } x_{2,s} + 2r}{2}\right) \leq 0,$$

where the equality holds at some time instant $t \geq 0$ if and only if $x_{2,s}(t) = 0$. By LaSalle's invariance principle [14] that $\lim_{t \rightarrow \infty} x_{1,s}(t) = 0$ and $\lim_{t \rightarrow \infty} x_{2,s}(t) = 0$ when $g'(t) \equiv 0$. Because of the particular structure of $(x_{1,s}, x_{2,s})$ -subsystem (29), by using [15, Lemma 3.6], one can show that any globally asymptotically stabilizing feedback for $(x_{1,s}, x_{2,s})$ -subsystem (29) when $g'(t) \equiv 0$ achieves global asymptotic stability of $(x_{1,s}, x_{2,s})$ -subsystem (29) when $\lim_{t \rightarrow \infty} g'(t) = 0$. Therefore, based on Part 1, $\lim_{t \rightarrow \infty} x_{1,s}(t) = 0$ and $\lim_{t \rightarrow \infty} x_{2,s}(t) = 0$.

Part 3. Combining the two parts above, we have $\lim_{t \rightarrow \infty} \|\mathbf{x}_s(t)\| = 0$. For the $(x_{1,s}, x_{2,s})$ -subsystem and $(x'_{3,s}, x'_{4,s})$ -subsystem, $\|\mathbf{x}_s(t)\|$ is bounded in any finite time. With the obtained result $\lim_{t \rightarrow \infty} \|\mathbf{x}_s(t)\| = 0$, we have $\|\mathbf{x}_s(t)\|$ is bounded on $[0, \infty)$.

References

- [1] C.-J. Wan, D.S. Bernstein, V.T. Coppola, Global stabilization of the oscillating eccentric rotor, *Nonlinear Dynamics* 10 (1996) 49–62, <http://dx.doi.org/10.1007/BF00114798>.
- [2] J. Zhao, I. Kanellakopoulos, Flexible backstepping design for tracking and disturbance attenuation, *International Journal of Robust and Nonlinear Control* 8 (1998) 331–348. DOI: 10.1002/(SICI)1099-1239(19980415/30)8:4/5 < 331::AID-RNC358 > 3.0.CO;2-A.
- [3] Z.P. Jiang, D.J. Hill, Y. Guo, Stabilization and tracking via output feedback for a nonlinear benchmark system, *Automatica* 34 (1998) 907–915, [http://dx.doi.org/10.1016/S0005-1098\(98\)00022-3](http://dx.doi.org/10.1016/S0005-1098(98)00022-3).
- [4] Z.P. Jiang, I. Kanellakopoulos, Global output-feedback tracking for a benchmark nonlinear system, *IEEE Transactions on Automatic Control* 45 (2000) 1023–1027, <http://dx.doi.org/10.1109/9.855577>.
- [5] Z. Petres, P. Baranyi, P. Korondi, H. Hashimoto, Trajectory tracking by TP model transformation: case study of a benchmark problem, *IEEE Transactions on Industrial Electronics* 54 (2007) 1654–1663, <http://dx.doi.org/10.1109/TIE.2007.894697>.
- [6] J. Huang, G. Hu, Control design for the nonlinear benchmark problem via the output regulation method, *Journal of Control Theory and Applications* 2 (2004) 11–19, <http://dx.doi.org/10.1007/s11768-004-0018-6>.
- [7] A. Pavlov, B. Janssen, N. van de Wouw, H. Nijmeijer, Experimental output regulation for a nonlinear benchmark system, *IEEE Transactions on Control Systems Technology* 15 (2007) 786–793, <http://dx.doi.org/10.1109/TCST.2006.890294>.
- [8] C. Fabio, Output regulation for the TORA benchmark via rotational position feedback, *Automatica* 47 (2011) 584–590, <http://dx.doi.org/10.1016/j.automatica.2011.01.008>.
- [9] W. Lan, B.M. Chen, Z. Ding, Adaptive estimation and rejection of unknown sinusoidal disturbances through measurement feedback for a class of non-minimum phase non-linear MIMO systems, *International Journal of Adaptive Control and Signal Processing* 20 (2006) 77–97, <http://dx.doi.org/10.1002/acs.888>.
- [10] Y. Jiang, J. Huang, Output regulation for a class of weakly minimum phase systems and its application to a nonlinear benchmark system. *American Control Conference*, St. Louis, MO, June 2009, pp. 5321–5326.
- [11] A. Fradkov, O. Tomchina, D. Tomchin, Controlled passage through resonance in mechanical systems, *Journal of Sound and Vibration* 330 (2011) 1065–1073, <http://dx.doi.org/10.1016/j.jsv.2010.09.031>.
- [12] Q. Quan, K.-Y. Cai, Additive decomposition and its applications to internal-model-based tracking, *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, December 2009, pp. 817–822.
- [13] C. Cao, N. Hovakimyan, Design and analysis of a novel $\mathcal{L}_1$ adaptive control architecture with guaranteed transient performance, *IEEE Transactions on Automatic Control* 53 (2008) 586–591, <http://dx.doi.org/10.1109/TAC.2007.914282>.
- [14] Hassan K. Khalil, *Nonlinear Systems*, Prentice-Hall, Upper Saddle River, New York, 2002.
- [15] H.J. Sussmann, E.D. Sontag, Y. Yang, A general result on the stabilization of linear systems using bounded controls, *IEEE Transactions on Automatic Control* 39 (1994) 2411–2425, <http://dx.doi.org/10.1109/9.362853>.
- [16] Q. Quan, K.-Y. Cai, Additive-output-decomposition-based dynamic inversion tracking control for a class of uncertain linear time-invariant systems, *The 51st IEEE Conference on Decision and Control*, Maui, HI, USA, December 2012, pp. 2866–2871.
- [17] Z. Petres, Polytopic Decomposition of Linear Parameter-Varying Models by Tensor-Product Model Transformation, PhD Dissertation, Doctorate School of Computer Science and Information Technology, Budapest, University of Technology and Economics, Budapest, Hungary, November 2006.
- [18] Z. Petres, P. Baranyi, H. Hashimoto, Approximation and complexity trade-off by TP model transformation in controller design: a case study of the TORA system, *Asian Journal of Control* 12 (2010) 575–585, <http://dx.doi.org/10.1002/asjc.222>.
- [19] M. Morari, E. Zafriou, *Robust Process Control*, 1st ed. Prentice-Hall, Upper Saddle River, NJ, 1989.
- [20] Q. Quan, H. Lin, K.-Y. Cai, Output feedback tracking control by additive state decomposition for a class of uncertain systems. *International Journal of Systems Science* (2013), <http://dx.doi.org/10.1080/00207721.2012.757379>.
- [21] Q. Quan, K.-Y. Cai, H. Lin, Additive-state-decomposition-based output feedback tracking control for systems with measurable nonlinearities and unknown disturbances (<http://arxiv.org/abs/1109.4516>).
- [22] Q. Quan, K.-Y. Cai, Additive-state-decomposition-based dynamic inversion stabilized control for a class of uncertain systems and its application to missile longitudinal tracking (<http://arxiv.org/abs/1211.6821>).
- [23] M. Jankovic, D. Fontaine, P.V. Kokotovic, TORA example: cascade and passivity-based control designs, *IEEE Transactions on Control Systems Technology* 4 (1996) 292–297, <http://dx.doi.org/10.1109/87.491203>.
- [24] L.-C. Hung, H.-P. Lin, H.-Y. Chung, Design of self-tuning fuzzy sliding mode control for TORA system, *Expert Systems with Applications* 32 (2007) 201–212, <http://dx.doi.org/10.1016/j.eswa.2005.11.008>.
- [25] S. Nazrulla, H.K. Khalil, Robust stabilization of non-minimum phase nonlinear systems using extended high-gain observers, *IEEE Transactions on Automatic Control* 56 (2011) 802–813, <http://dx.doi.org/10.1109/ACC.2008.4586742>.