INTERNATIONAL JOURNAL OF ADAPTIVE CONTROL AND SIGNAL PROCESSING
Int. J. Adapt. Control Signal Process. 2013; **27**:400–412
Published online 27 June 2012 in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/acs.2315

Internal-model-based control to reject an external signal generated by a class of infinite-dimensional systems

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SUMMARY

In this paper, an *internal-model-based controller* (IMBC) is designed to reject an external signal generated by a class of infinite-dimensional systems for a class of nonlinear systems. First, a new description of the external signal is proposed. By this new description, the IMBC is designed, and the resulting closed-loop error dynamics are analyzed with the help of a Lyapunov–Krasovskii functional. Compared with existing IMBCs, the designed IMBC provides the flexibility to cope with more types of external signals as well as to choose controller parameters to achieve a tradeoff between rejection performance and stability. Finally, the method is applied to attitude control of a quadrotor aircraft. Effectiveness of the IMBC is demonstrated by simulation. Copyright © 2012 John Wiley & Sons, Ltd.

Received 2 January 2010; Revised 15 November 2010; Accepted 4 June 2012

KEY WORDS: infinite-dimensional systems; internal model; nonlinear systems; rejection; repetitive control

1. INTRODUCTION

The *internal-model-based control* (IMBC, or *internal-model-based controller*, also designated IMBC) has received considerable attention over these years [1–4]. The external signals considered therein are usually assumed to be generated by finite-dimensional systems. However, many external signals, such as the triangular wave, are only generated by infinite-dimensional systems. For such external signals, there has been little research on designing IMBCs for linear systems [5–7], let alone designing IMBCs for nonlinear systems. This motivates us to design an IMBC for a class of nonlinear systems to reject an external signal generated by a class of infinite-dimensional systems [8, 9]. There exist two major difficulties.

(i) It is difficult to design the IMBC by frequency-domain methods. The internal model principle [10] reveals that asymptotic tracking (or rejection) of the external signal can be achieved by incorporating a copy of the exogenous dynamics, namely the infinite-dimensional system, in the feedback loop. When the infinite-dimensional system and its copy reduce to systems as $w(t) = w(t - T)$, the resulting closed-loop system becomes a repetitive control system correspondingly [11]. In other words, a repetitive control system is a special case of the resulting closed-loop system. It was proved in [11] that, for a class of general linear plants, exponential stability of repetitive control systems could be achieved only when the plant is proper but not strictly proper. For this reason, as shown in Figure 1, a filtered repetitive controller[‡] was further

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‡In this paper, we have replaced the term modified' in [11] with the more descriptive term 'filtered'.

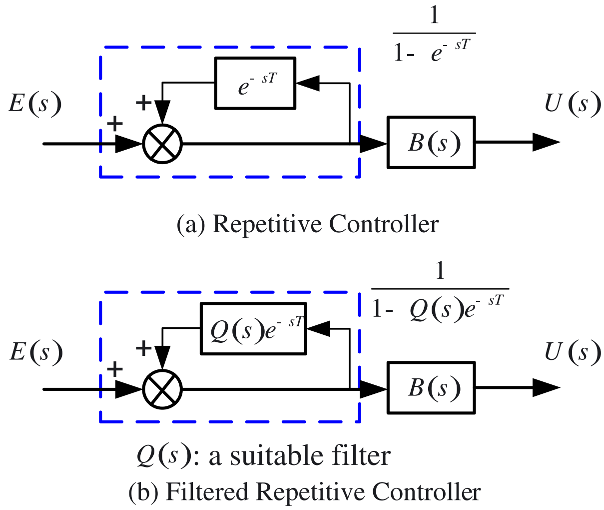

Figure 1. A suitable filter $Q(s)$ is introduced into a repetitive controller to form a filtered repetitive controller.

proposed to achieve a tradeoff between the tracking (or rejection) performance and stability [11, 12]. With the considered external signals, we also need to consider a tradeoff between the tracking (or rejection) performance and stability in the resulting closed-loop system when designing the IMBC. Unfortunately, it is not trivial to follow the idea of the filtered repetitive controller because the related theories are derived in the frequency domain and can be applied only with difficulty, if at all, to nonlinear systems.

(ii) It is difficult to design the IMBC by the geometric approach. The geometric approach [1],[2, pp. 10, 43–44],[10, 13] is often used to design IMBCs. However, the existing theories on geometric approach are only applicable to the case where the closed-loop system is finite dimensional. Whereas, when the external signal is generated by an infinite-dimensional system, the resulting closed-loop system, which contains the copy, is infinite dimensional as well.

To overcome the two major difficulties earlier, a new description of the external signal is proposed, and the resulting closed-loop error dynamics are analyzed with the help of a Lyapunov–Krasovskii functional. The contributions of this paper are (i) for a class of nonlinear systems, a novel method to design an IMBC to reject an external signal generated by an infinite-dimensional system; (ii) a tradeoff achieved by tuning controller parameters between rejection performance and stability; (iii) a new model proposed to describe a signal generated by an infinite-dimensional system.

We use the following notation. $\mathbb{R}^n$ is Euclidean space of dimension n, and $\mathbb{R}^+$ denotes the space of nonnegative reals in $\mathbb{R}$. $\|\cdot\|$ denotes the Euclidean vector norm or induced matrix norm. $\mathcal{C}([-T,0];\mathbb{R}^n)$ denotes the space of continuous n-dimensional vector functions on $[-T,0]$. $\|x_t\|_c \triangleq \sup_{\theta\in[-T,0]} \|x(t+\theta)\|$, where $x_t \triangleq x_t(\theta) = x(t+\theta)$, $\theta \in [-T,0]$. $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote the minimum and maximum eigenvalues, respectively, of a symmetric matrix X. $X^\top$ is used for the transpose of matrix X. $X > 0 (X \geqslant 0, X < 0, X \leqslant 0)$ denotes matrix X is a positive definite (positive semidefinite, negative definite, negative semidefinite) matrix. I_n is the identity matrix with dimension n. $0_{n_1\times n_2}$ denotes a zero matrix with dimension $n_1 \times n_2$.

2. EXTERNAL SIGNAL

The external signal is assumed to be generated by a class of infinite-dimensional systems. In this section, we introduce the class of systems and propose a new model to describe the external signal.

DOI: 10.1002/acs

2.1. *A class of infinite-dimensional systems*

Before introducing the class of systems, we define

$$\mathcal{D}y(t) \triangleq \begin{bmatrix} y_1(t) \\ \dot{y}_2(t) \end{bmatrix}, \mathcal{F}y(t) \triangleq \begin{bmatrix} y_1(t-T) \\ y_2(t) \end{bmatrix}$$

where $y(t) = \begin{bmatrix} y_1^\top(t) & y_2^\top(t) \end{bmatrix}^\top \in \mathbb{R}^{p_1+p_2}$, $y_1(t) \in \mathbb{R}^{p_1}$ and $y_2(t) \in \mathbb{R}^{p_2}$. '$\cdot$' represents the right-hand derivative here. For simplicity, let $p = p_1 + p_2$.

Assumption 1
The external signal $w(t)$ is generated by a class of infinite-dimensional systems as follows:

$$\begin{aligned} \mathcal{D}x_w(t) =& \mathcal{F}A_w x_w(t) \\ w(t) =& C_w x_w(t), x_w(\theta) = \varphi(\theta), \theta \in [-T, 0] \end{aligned} \tag{1}$$

where $x_w(t) \in \mathbb{R}^p$, $A_w \in \mathbb{R}^{p\times p}$, and $C_w \in \mathbb{R}^{m\times p}$. Moreover, $\sup_{t\in[0,\infty)} \|x_w(t)\| < M_1 < \infty$ and $\sup_{t\in[0,\infty)} \|\dot{x}_w(t)\| < M_2 < \infty$, where $M_1, M_2 \in \mathbb{R}^+$.

Example 1 shows that the system (1) can represent not only a finite-dimensional system but also an infinite-dimensional system.

Example 1
For simplicity, let

$$\begin{aligned} A_w =& \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, A_1 \in \mathbb{R}^{p_1\times p_1}, A_2 \in \mathbb{R}^{p_1\times p_2}, A_3 \in \mathbb{R}^{p_2\times p_1}, A_4 \in \mathbb{R}^{p_2\times p_2}, \\ C_w =& \begin{bmatrix} C_1 & C_2 \end{bmatrix}, C_1 \in \mathbb{R}^{m\times p_1}, C_2 \in \mathbb{R}^{m\times p_2}. \end{aligned}$$

If $p_1 = 0$, then the system (1) reduces to a finite-dimensional system $\dot{x}_w(t) = A_4 x_w(t)$, $w(t) = C_2 x_w(t)$. If $p_2 = 0$, then the system (1) reduces to an infinite-dimensional system $x_w(t) = A_1 x_w(t-T)$, $w(t) = C_1 x_w(t)$. In particular, if $p_1 = m$, $A_1 = I_m$, and $C_1 = I_m$, then (1) becomes $w(t) = w(t-T)$, which can generate any T-periodic signal vector. If $p_1 = p_2 = m$, $A_1 = C_1 = 0_{m\times m}$, and $A_2 = C_2 = I_m$, then the system (1) reduces to a retarded system $\dot{w}(t) = A_4 w(t) + A_3 w(t-T)$.

2.2. *New description*

It is well known that, with an appropriate initial condition, any step signal or any T-periodic signal can be generated by an autonomous system, namely $\dot{w}(t) = 0$ or $w(t) = w(t-T)$, respectively. In fact, both types of signal can be described by nonautonomous systems as well. It is easy to verified that, any step signal can also be generated by $\dot{w}(t) = -\alpha w(t) + \sigma(t)$, where $\sigma(t) = \alpha w(t)$ is viewed as a signal depending on t, and any T-periodic signal satisfying Assumption 1 can be generated by $\alpha\dot{w}(t) = -w(t) + (1-\alpha)w(t-T) + \sigma(t)$, where $\sigma(t) = \alpha[w(t-T) + \dot{w}(t)]$ is viewed as a signal depending on t. In particular, by setting $\alpha = 0$, the descriptions in form of nonautonomous systems reduce to those in form of autonomous systems. With the freedom of α, the former is more general. This will help to design an IMBC that can recover a filtered repetitive controller. In the following, a new description of the external signal generated by (1) is proposed. Before proceeding further, we introduce the following system:

$$\begin{aligned} A_{w_0}\dot{x}_v(t) =& A_{w_1}x_v(t) + A_{w_2}x_v(t-T) + \sigma(t) \\ v(t) =& C_w x_v(t), x_v(\theta) = \varphi(\theta), \theta \in [-T, 0] \end{aligned} \tag{2}$$

where $x_v(t) \in \mathbb{R}^p$ and $A_{w_0}, A_{w_1}, A_{w_2} \in \mathbb{R}^{p\times p}$.

DOI: 10.1002/acs

Lemma 1
Under Assumption 1, $\left(x_w^*, w\right)$ is the solution of (1). Then, there exists a bounded function σ such that the solution $\left(x_v^*, v\right)$ of (2) satisfies

$$x_v^* = x_w^*, v = w \tag{3}$$

where σ can be chosen as follows:

$$\sigma(t) = A_{w0}\dot{x}_w^*(t) - \mathcal{D}x_w^*(t) - \mathcal{F}A_w x_w^*(t) - A_{w1}x_w^*(t) - A_{w2}x_w^*(t-T). \tag{4}$$

Proof
Under Assumption 1, because $\left(x_w^*, w\right)$ is the solution of (1), thus

$$\begin{aligned} \mathcal{D}x_w^*(t) =& \mathcal{F}A_w x_w^*(t) \\ w(t) =& C_w x_w^*(t), x_w^*(\theta) = \varphi(\theta), \theta \in [-T, 0]. \end{aligned}$$

The aforementioned system can be rewritten as

$$\begin{aligned} A_{w0}\dot{x}_w^*(t) =& A_{w1}x_w^*(t) + A_{w2}x_w^*(t-T) + \sigma(t) \\ w(t) =& C_w x_w^*(t), x_w^*(\theta) = \varphi(\theta), \theta \in [-T, 0] \end{aligned} \tag{5}$$

where $\sigma(t)$ is defined in (4). Therefore, $\left(x_w^*, w\right)$ is also a solution of the system (2) when the system is driven by the $\sigma(t)$ defined in (4). By the uniqueness of solutions, there exists such a function σ that Equation (3) holds. Because $\sup\limits_{t\in[0,\infty)} \|x_w(t)\| < M_1 < \infty$ and $\sup\limits_{t\in[0,\infty)} \|\dot{x}_w(t)\| < M_2 < \infty$ by Assumption 1, the function σ is bounded. □

From Lemma 1, $w(t)$ defined in (1) can also be generated by

$$\begin{aligned} A_{w0}\dot{x}_w(t) =& A_{w1}x_w(t) + A_{w2}x_w(t-T) + \sigma(t) \\ w(t) =& C_w x_w(t) \end{aligned} \tag{6}$$

with an appropriate initial condition and a bounded function σ. In particular, the new description (6) can reduce to (1).

Examples 2 and 3 are given to show the effectiveness of Lemma 1.

Example 2
If $p_1 = 0$, $p_2 = 3$, and A_w, C_w in the system (1) are

$$A_w = 0_{3\times1}, C_w = I_3 \tag{7}$$

then the system (1) reduces to

$$\dot{x}_w(t) = 0_{3\times1}, w(t) = x_w(t) \tag{8}$$

where $x_w(t) \in \mathbb{R}^3$. Then, the external signal $w(t)$ represents a vector of step signals. From Lemma 1, any external signal $w(t)$ generated by the system (1) can also be generated by (6) with

$$A_{w_0} = I_3 \in \mathbb{R}^{3\times3}, A_{w_1} = A_w - \alpha I_3 \in \mathbb{R}^{3\times3}, A_{w_2} = 0_{3\times3} \tag{9}$$

and a bounded function σ, where $\alpha \in \mathbb{R}$. By setting $\alpha = 0$, then $\sigma(t) \equiv 0$ and the new description (6) with (9) reduces to (8).

Example 3
If $p_1 = 3$, $p_2 = 0$, and A_w, C_w in the system (1) are

$$A_w = I_3, C_w = I_3 \tag{10}$$

then the system (1) reduces to

$$w(t) = w(t-T). \tag{11}$$

The external signal $w(t)$ can represent any T-periodic signal. If A_{w_0}, A_{w_1}, and A_{w_2} are chosen to be

$$A_{w_0} = \alpha I_3, A_{w_1} = -I_3, A_{w_2} = (1-\alpha) I_3, \tag{12}$$

where $\alpha \in \mathbb{R}$, then by Lemma 1, the system (6) is simplified to

$$\alpha \dot{w}(t) = -w(t) + (1-\alpha) w(t-T) + \sigma(t). \tag{13}$$

By setting $\alpha = 0$, then $\sigma(t) \equiv 0$ and (13) reduces to (11).

The new description (6) is a retarded system driven by an external signal $\sigma(t)$, $t \in [0,\infty)$. The retarded system can be designed appropriately to achieve a tradeoff between its stability and the bound on $\|\sigma(t)\|$, $t \in [0,\infty)$. In Examples 2 and 3, setting $\alpha = 0$ results in $\sigma(t) \equiv 0$, but the resulting systems are marginally stable, whereas with $\alpha > 0$, the retarded systems are asymptotically stable, although $\sup_{t\in[0,\infty)} \|\sigma(t)\| > 0$. We next examine the relationship between $\sigma(t)$ and α in Example 3 by the frequency response method. The Laplace transform of (13) results in $\sigma(s) = G(s)w(s)$, where $\sigma(s)$ and $w(s)$ are the Laplace transforms of $\sigma(t)$ and $w(t)$, respectively; $G(s) = \alpha s + 1 - (1-\alpha)\exp(-sT)$. The frequency responses of $G(s)$ with $T = 2\pi$ and $\alpha = 0.1, 0.05, 0.01$ are shown in Figure 2.

As shown in Figure 2, $G(s)$ has a comb shape with notches matching the frequencies of the periodic signal $w(t)$. This makes the frequency response of $G(s)$ close to zero at $k, k = 0, 1, 2, \cdots$. As the parameter α decreases, the periodic components, especially in low frequency band, will be attenuated by $G(s)$ more strongly. Since low frequency band is dominant in most periodic signals, this will result in a smaller upper bound on $\|\sigma(t)\|$. In particular, $\sigma(t) \equiv 0$ when $\alpha = 0$.

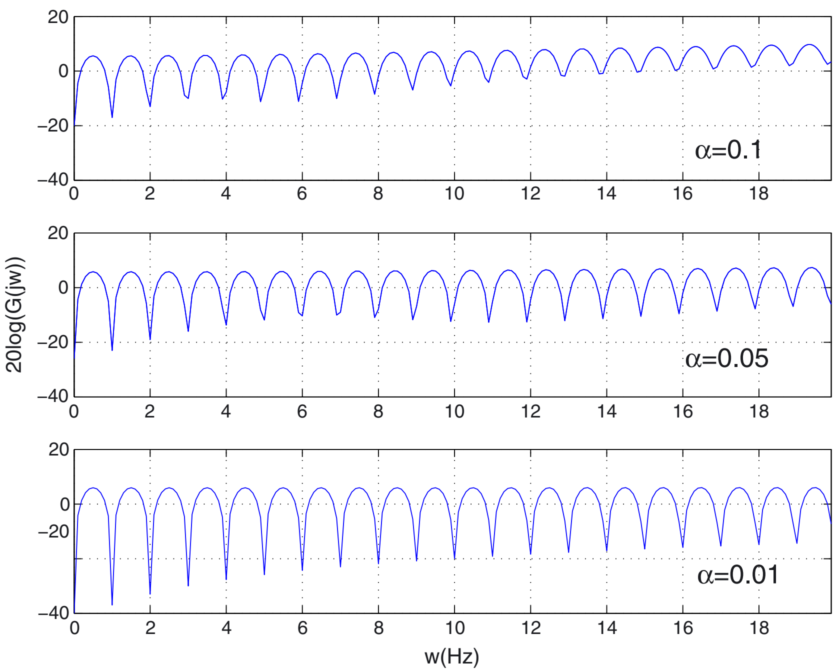

Figure 2. Frequency responses of $G(s)$.

DOI: 10.1002/acs

3. PROBLEM FORMULATION

To illustrate the generality of the IMBC, we consider the following nonlinear error dynamics [14, 15]:

$$\dot{e}(t) = f(t, e(t)) + b(t, e(t))\,[w(t) - \hat{w}(t)] \tag{14}$$

where $e(t) \in \mathbb{R}^n$ is the error, $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $b : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$; $w(t) \in \mathbb{R}^m$ is an external signal satisfying Assumption 1; $\hat{w}(t) \in \mathbb{R}^m$ is an estimate of $w(t)$ to be specified.

We impose the following assumption on the system (14).

Assumption 2
For $\dot{e}(t) = f(t, e(t))$, there exists a known function $V_0 : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ that satisfies the following inequalities

$$\begin{aligned} c_1 \|e(t)\|^2 \leqslant V_0(t, e(t)) \leqslant &\quad c_2 \|e(t)\|^2 \\ \frac{\partial V_0}{\partial t} + \frac{\partial V_0}{\partial e} f(t, e(t)) \leqslant & \ -c_3 \|e(t)\|^2 \end{aligned} \tag{15}$$

where $c_1, c_2,$ and c_3 are all positive reals.

Under Assumptions 1 and 2, the control objective is to seek an IMBC, which generates $\hat{w}(t)$, to reject the external signal $w(t)$.

Remark 1
Note that a sine wave and a triangular wave are bounded; moreover, their right-hand derivatives exist and are also bounded; hence, Assumption 1 is satisfied when $w(t)$ is a sine wave or a triangular wave. If the zero solution of $\dot{e}(t) = f(t, e(t))$ is globally exponentially stable, then Assumption 2 is satisfied. As in [16] and [17], exponentially stable controllers together with the considered plants therein can result in the error dynamics as $\dot{e}(t) = f(t, e(t))$ and make Assumption 2 hold.

4. INTERNAL-MODEL-BASED CONTROLLER DESIGN AND STABILITY ANALYSIS

In this section, the IMBC is designed by using the new description (6). Stability of the resulting closed-loop error dynamics is then analyzed with the help of a Lyapunov–Krasovskii functional. These results are stated in Theorem 1.

4.1. *Internal-model-based controller design*

By the new description (6), we design the IMBC as follows:

$$\begin{aligned} A_{w_0}\dot{\hat{x}}_w(t) =& A_{w_1}\hat{x}_w(t) + A_{w_2}\hat{x}_w(t-T) + H(t, e(t)) \\ \hat{w}(t) =& C_w \hat{x}_w(t), \hat{x}_w(\theta) = 0, \theta \in [-T, 0] \end{aligned} \tag{16}$$

where $H(t, e(t)) \in \mathbb{R}^p$ will be designed later and $\hat{x}_w(t) \in \mathbb{R}^p$. Subtracting (16) from (6) yields

$$\begin{aligned} A_{w_0}\dot{\tilde{x}}_w(t) =& A_{w_1}\tilde{x}_w(t) + A_{w_2}\tilde{x}_w(t-T) - H(t, e(t)) + \sigma(t) \\ \tilde{w}(t) =& C_w \tilde{x}_w(t) \end{aligned} \tag{17}$$

where $\tilde{x}_w \triangleq x_w - \hat{x}_w$ and $\tilde{w} \triangleq w - \hat{w}$. In (17), the initial condition on $\tilde{x}_w$ is bounded. We do not concern ourselves with the concrete value of the initial condition as the following results hold globally. Combining (14) and (17) yields the closed-loop error dynamics as follows:

$$E\dot{z}(t) = f_a(t, z(t)) + f_d(z(t-T)) + b_a(t, z(t))\sigma(t) \tag{18}$$

with $z = \begin{bmatrix} \tilde{x}_w \\ e \end{bmatrix} \in \mathbb{R}^{p+n}$, $f_a(t, z(t)) = \begin{bmatrix} A_{w_1}\tilde{x}_w(t) - H(t, e(t)) \\ f(t, e(t)) + b(t, e(t))C_w\tilde{x}_w(t) \end{bmatrix} \in \mathbb{R}^{p+n}$, $f_d(z(t-T)) = \begin{bmatrix} A_{w_2}\tilde{x}_w(t-T) \\ 0_{n\times 1} \end{bmatrix} \in \mathbb{R}^{p+n}$, $E = \begin{bmatrix} A_{w_0} & 0_{p\times n} \\ 0_{n\times p} & I_n \end{bmatrix} \in \mathbb{R}^{(p+n)\times(p+n)}$, $b_a(t, z(t)) = \begin{bmatrix} I_p \\ 0_{n\times p} \end{bmatrix} \in \mathbb{R}^{(p+n)\times p}$.

DOI: 10.1002/acs

4.2. Stability analysis

To begin with, the following result is needed.

Definition 1 ([18]*)*
A solution $z(t, z_{t_0})$ of (18) is said to be uniformly ultimately bounded with respect to the bound ϵ if for each $\delta > 0$, there exists $T' = T'(\epsilon, \delta) > 0$ independent of $t_0 \geqslant 0$ such that $\left\| z(t, z_{t_0}) \right\| \leqslant \epsilon$ for all $t \geqslant t_0 + T'$ when $\left\| z_{t_0} \right\|_c < \delta$, where $z_{t_0} \in \mathcal{C}([-T, 0]; \mathbb{R}^n)$.

Lemma 2 ([18]*)*
Assume that there exists a continuously differentiable functional $V(t, z_t) : \mathbb{R}^+ \times \mathcal{C}([-T, 0]; \mathbb{R}^n) \to \mathbb{R}^+$ such that

$$\gamma_1 \|z(t)\|^2 \leqslant V(t, z_t) \leqslant \gamma_2 \|z(t)\|^2 + \int_{t-T}^{t} \|z(s)\|^2 \mathrm{d}s \tag{19}$$

where γ_1 and γ_2 are positive reals. If there exists $\delta_0 \in \mathbb{R}^+$ such that

$$\dot{V}(t, z_t) \leqslant -\gamma_3 \|z(t)\|^2 + \delta_0 \tag{20}$$

where γ_3 is a positive real, then the solutions of (18) are uniformly ultimately bounded with respect to the bound $\sqrt{(1+\epsilon_1)\frac{\delta_0}{\gamma_1\gamma_3}(\gamma_2 + T)}$, where ϵ_1 is an arbitrarily small positive real.

Theorem 1
Suppose (i) Assumptions 1 and 2 hold; (ii) in (16), $0 \leqslant A_{w_0} = A_{w_0}^{\top} \in \mathbb{R}^{p \times p}$; moreover, there exist a $\mu \in \mathbb{R}^+$ and a positive real ε such that

$$\Omega + \varepsilon\sigma_0 L L^{\top} \leqslant 0 \tag{21}$$

where

$$\Omega = \begin{bmatrix} A_{w_1} + A_{w_1}^{\top} + \mu I_p & A_{w_2} \\ A_{w_2}^{\top} & -\mu I_p \end{bmatrix}, L = \begin{bmatrix} I_p & 0_{p\times p} \end{bmatrix}^{\top} \text{ and } \sigma_0 = \sup_{t\in[0,\infty)} \|\sigma(t)\|;$$

(iii) the estimate $\hat{w}(t)$ in the system (14) is generated by (16) with

$$H(t, e(t)) = C_w^{\top} b^{\top}(t, e(t)) \frac{\partial V_0}{\partial e} \tag{22}$$

and $A_{w_0}, A_{w_1}, A_{w_1}$ satisfying (21). We claim that (i) if $\sigma_0 = 0$, then $e \in \mathcal{L}_\infty[0,\infty) \cap \mathcal{L}_2[0,\infty)$; (ii) if $A_{w_0} > 0$, $\mu > 0$, and $\sigma_0 > 0$, then for any bounded initial condition, $e(t)$ in (18) is uniformly ultimately bounded with respect to the bound ϵ defined in (29).

Proof
For (18), design a Lyapunov–Krasovskii functional $W(t, z_t)$ as

$$W(t, z_t) = 2V_0(t, e(t)) + \tilde{x}_w^{\top}(t) A_{w_0} \tilde{x}_w(t) + \mu \int_{t-T}^{t} \tilde{x}_w^{\top}(s)\tilde{x}_w(s)\mathrm{d}s \tag{23}$$

where V_0 satisfies Assumption 2, $0 \leqslant A_{w_0} = A_{w_0}^{\top} \in \mathbb{R}^{p\times p}$ and $\mu \in \mathbb{R}^+$.

Taking the time derivative of $W(t, z_t)$ along (18) yields

$$\begin{aligned}\dot{W}(t, z_t) =& 2\frac{\partial V_0}{\partial t} + 2\left(\frac{\partial V_0}{\partial e}\right)^{\top} f(t, e(t)) + 2\left(\frac{\partial V_0}{\partial e}\right)^{\top} b(t, e(t)) C_w \tilde{x}_w(t) \\ & - 2\tilde{x}_w^{\top}(t) H(t, e(t)) + \bar{x}_w^{\top}(t)\Omega\bar{x}_w(t) + 2\tilde{x}_w^{\top}(t)\sigma(t)\end{aligned} \tag{24}$$

where $\bar{x}_w(t) = \begin{bmatrix} \tilde{x}_w^{\top}(t) & \tilde{x}_w^{\top}(t-T) \end{bmatrix}^{\top} \in \mathbb{R}^{2p}$.

DOI: 10.1002/acs

Since $\Omega \leqslant -\varepsilon\sigma_0 LL^\top$ by (21), we obtain $\bar{x}_w^\top(t)\Omega\bar{x}_w(t) \leqslant -\varepsilon\sigma_0 \|\tilde{x}_w(t)\|^2$. Then, further using Assumption 2 and (22), we have

$$\dot{W}(t, z_t) \leqslant -2c_3 \|e(t)\|^2 - \varepsilon\sigma_0 \|\tilde{x}_w(t)\|^2 + 2\sigma_0 \|\tilde{x}_w(t)\|. \quad (25)$$

On the basis of the aforementioned results, we now prove conclusions (i) and (ii).

(i) If $\sigma_0 = 0$, then (25) becomes

$$\dot{W}(t, z_t) \leqslant -2c_3 \|e(t)\|^2. \quad (26)$$

From the aforementioned inequality, we obtain

$$W(t, z_t) \leqslant W(0, z_0). \quad (27)$$

Integrating both sides of (26) yields

$$W(t, z_t) - W(0, z_0) \leqslant -2c_3 \int_0^t \|e(s)\|^2 \,\mathrm{d}s.$$

Note that $W(t, z_t) \geqslant 0$ and $c_3 > 0$, the aforementioned inequality becomes

$$\int_0^t \|e(s)\|^2 \,\mathrm{d}s \leqslant \frac{1}{2c_3} W(0, z_0). \quad (28)$$

Because $W(t, z_t) \geqslant 2V_0(t, e(t)) \geqslant c_1 \|e(t)\|^2$ by Assumption 2, hence $e \in \mathcal{L}_\infty[0, \infty) \cap \mathcal{L}_2[0, \infty)$ by (27) and (28).

(ii) For all positive reals ε and σ_0, we have

$$-\frac{\varepsilon}{2}\sigma_0 \|\tilde{x}_w(t)\|^2 + 2\sigma_0 \|\tilde{x}_w(t)\| - \frac{2}{\varepsilon}\sigma_0 \leqslant 0$$

then (25) becomes

$$\dot{W}(t, z_t) \leqslant -2c_3 \|e(t)\|^2 - \frac{\varepsilon}{2}\sigma_0 \|\tilde{x}_w(t)\|^2 + \frac{2}{\varepsilon}\sigma_0.$$

With $\mu > 0$, let $V(t, z_t) = \frac{1}{\mu} W(t, z_t)$. Note that $\|z(t)\|^2 = \|e(t)\|^2 + \|\tilde{x}_w(t)\|^2$, then the inequality (19) is satisfied with $\gamma_1 = \frac{1}{\mu}\min(2c_1, \lambda_{\min}(A_{w_0}))$ and $\gamma_2 = \frac{1}{\mu}\max(2c_2, \lambda_{\max}(A_{w_0}))$. Moreover, the inequality (20) is satisfied with $\gamma_3 = \frac{1}{\mu}\min\left(2c_3, \frac{\varepsilon}{2}\sigma_0\right)$ and $\delta_0 = \frac{2}{\mu\varepsilon}\sigma_0$. Because $A_{w_0} > 0$ and $\sigma_0 > 0$, hence $\gamma_1, \gamma_2, \gamma_3, \delta_0$ are all positive reals. Note the fact that $\|e(t)\| \leqslant \|z(t)\|$ by the definition of $z(t)$, then $e(t)$ in (18) is uniformly ultimately bounded with respect to the bound

$$\epsilon = \sqrt{\frac{2(1+\epsilon_1)\sigma_0\left(\max(2c_2, \lambda_{\max}(A_{w_0})) + \mu T\right)}{\varepsilon \min(2c_1, \lambda_{\min}(A_{w_0}))\min\left(2c_3, \frac{\varepsilon}{2}\sigma_0\right)}} \quad (29)$$

by Lemma 2, where ϵ_1 is an arbitrarily small positive real.

□

Remark 2

Obviously, the time-domain analysis proposed in this paper can also be applicable to linear systems. Compared with the frequency-domain analysis, the proposed analysis can give some performance specifications in the time domain, such as the ultimate bound.

5. ATTITUDE CONTROL OF A QUADROTOR AIRCRAFT

To show its effectiveness, we apply the proposed method to attitude control of a quadrotor aircraft subject to disturbances.

5.1. Error dynamics

The unit quaternion is a vector denoted by $(q_0\ q)$, where $q_0(t) \in \mathbb{R}$, $q(t) \in \mathbb{R}^3$ are the scalar part and vector part of the unit quaternion, respectively, and $q_0^2(t) + \|q(t)\|^2 = 1$. The unit quaternion, which is free of singularity, is used to represent the attitude kinematics of a quadrotor aircraft as follows [19]:

$$\dot{q}(t) = \frac{1}{2}[q(t) \times \omega(t) + q_0(t)\omega(t)] \tag{30}$$

$$\dot{q}_0(t) = -\frac{1}{2}q^\top(t)\omega(t) \tag{31}$$

where $\omega(t) \in \mathbb{R}^3$ denotes the angular velocity of the airframe in the body fixed frame. For simplicity, the dynamic equation of attitude motion is assumed to be

$$\dot{\omega}(t) = -J^{-1}\omega(t) \times J\omega(t) + J^{-1}\tau(t) + w(t) \tag{32}$$

where $J \in \mathbb{R}^{3\times3}$ is the inertial matrix, $\tau(t) \in \mathbb{R}^3$ is the control torque, and $w(t) \in \mathbb{R}^3$ is the disturbance vector. By the coordinate transformation $x = \omega + 2q$, the systems (30)–(32) is converted as follows:

$$\begin{aligned}
\dot{q}(t) &= \frac{1}{2}[q(t) \times x(t) + q_0(t)x(t)] - q_0(t)q(t) \\
\dot{q}_0(t) &= -\frac{1}{2}\omega^\top(t)x(t) + q(t) \\
\dot{x}(t) &= -J^{-1}\omega(t) \times J\omega(t) + J^{-1}\tau(t) + [q(t) \times x(t) + q_0(t)x(t) - 2q_0(t)q(t)] + w(t).
\end{aligned} \tag{33}$$

Design $\tau = J\left[J^{-1}\omega \times J\omega - (q \times x + q_0 x) + 2q_0 q - 2x - \hat{w}\right]$. Then,

$$\begin{aligned}
\dot{q}(t) &= \frac{1}{2}[q(t) \times x(t) + q_0(t)x(t)] - q_0(t)q(t) \\
\dot{q}_0(t) &= -\frac{1}{2}\omega^\top(t)x(t) + q(t) \\
\dot{x}(t) &= -2x(t) + w(t) - \hat{w}(t).
\end{aligned} \tag{34}$$

The aforementioned system can be rewritten in the form of (14) with

$$\begin{aligned}
e &= \begin{bmatrix} q^\top & x^\top \end{bmatrix}^\top, \\
f(t, e(t)) &= \begin{bmatrix} \frac{1}{2}(q(t) \times x(t) + q_0(t)x(t)) - q_0(t)q(t) \\ -2x(t) \end{bmatrix}, \\
b(t, e(t)) &= \begin{bmatrix} 0_{3\times3} \\ I_3 \end{bmatrix}.
\end{aligned}$$

Here, $q_0(t)$ is generated by $\dot{q}_0(t) = -\frac{1}{2}\omega^\top(t)x(t) + q(t)$.

5.2. Verification of Assumption 2

For the system $\dot{e}(t) = f(t, e(t))$, the Lyapunov function is chosen to be

$$V_0(t) = [1 - q_0(t)]^2 + q^\top(t)q(t) + x^\top(t)x(t).$$

The derivative of $V_0(t)$ along $\dot{e}(t) = f(t, e(t))$ is

$$\begin{aligned}
\dot{V}_0(t) &= \frac{\partial V_0}{\partial t} + \frac{\partial V_0}{\partial e} f(t, e(t)) \\
&= q^\top(t)[-2q(t) + x(t)] - 4x^\top(t)x(t) \\
&\leqslant -\|e(t)\|^2.
\end{aligned} \tag{35}$$

DOI: 10.1002/acs

Let $q^{\top}(t)q(t) = \sin^2 \frac{\theta}{2}$ and $q_0(t) = \cos \frac{\theta}{2}$, where $0 \leqslant \theta \leqslant \pi$ [2, pp. 198]. Then,

$$\begin{aligned}[1 - q_0(t)]^2 + q^{\top}(t)q(t) =& 2 - 2q_0(t)\\ =& 2 - 2\cos\frac{\theta}{2}\\ =& 4\sin^2\frac{\theta}{4}.\end{aligned}$$

Since $\sin^2 \frac{\theta}{4} \leqslant \sin^2 \frac{\theta}{2}$ when $0 \leqslant \theta \leqslant \pi$, we have $[1 - q_0(t)]^2 + q^{\top}(t)q(t) \leqslant 4\|q(t)\|^2$. Consequently,

$$\frac{1}{2}\|e(t)\|^2 \leqslant V_0(t) \leqslant 4\|e(t)\|^2. \tag{36}$$

From (36) and (35), Assumption 2 is satisfied.

5.3. Numerical simulation

The simulation parameters are chosen as follows: the inertial matrix J of a quadrotor aircraft is as in [19] that $J = \mathrm{diag}(0.16, 0.16, 0.32)$ kg·m^2. The initial condition of (30)–(32) is $q_0(0) = 0.707$, $q(0) = \begin{bmatrix} -0.4 & -0.3 & 0.5 \end{bmatrix}^T$, and $\omega(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ rad/s.

5.3.1. Case 1: w is a vector of step disturbances. The external signal $w(t) \equiv \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ N· m is assumed to be the vector of step disturbances, which can be generated by the system (1) with (7). Assumption 1 is satisfied. If A_{w_0}, A_{w_1}, and A_{w_2} are chosen as in (9), then Ω in (21) becomes

$$\Omega = \begin{bmatrix} (-2\alpha + \mu)I_3 & 0_{3\times3} \\ 0_{3\times3} & -\mu I_3 \end{bmatrix}.$$

If $\alpha = 0$, then $\sigma_0 = 0$ in the example. The inequality (21) is satisfied with $\mu = 0$. If $\alpha > 0$, then $\sigma_0 > 0$. The inequality (21) is satisfied with $\mu = \frac{\alpha}{2}$ and $\varepsilon = \frac{\alpha}{2\sigma_0}$. Then, the IMBC (16) can be written as

$$\begin{aligned}\dot{\hat{x}}_w(t) =& -\alpha\hat{x}_w(t) + 2x(t)\\ \hat{w}(t) =& \hat{x}_w(t), \hat{x}_w(\theta) = 0, \theta \in [-T, 0].\end{aligned} \tag{37}$$

Under Assumptions 1 and 2, we can conclude by Theorem 1 that if $\alpha = 0$, then $e \in \mathcal{L}_\infty[0,\infty) \cap \mathcal{L}_2[0,\infty)$ when the system (34) is driven by the IMBC (37). Furthermore, in this case, the function $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and the function $b : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ in (14) are bounded when e is bounded on $\mathbb{R}^+$. Then, $\dot{e} \in \mathcal{L}_\infty[0,\infty)$. This implies that $\lim_{t\to\infty} \|e(t)\| = 0$ by Barbalat's lemma [20]. Under Assumptions 1 and 2, if $\alpha > 0$, then the tracking error $e(t)$ is uniformly ultimately bounded. For rejection performance comparison, we define $E(t) = \|e(t)\|$. As shown in Figure 3, $E(t)$ approaches 0 as t increases with $\alpha = 0$ and $E(t)$ is bounded with $\alpha = 0.1$ or 0.05. These results are consistent with conclusion (i) and conclusion (ii) in Theorem 1, respectively.

5.3.2. Case 2: w is a vector of T-periodic disturbances. The external signal $w(t) = \begin{bmatrix}\sin(t) & \sin(t+1)\cos(t) & \sin^2(t+1)\end{bmatrix}^T$ N·m is assumed to be the vector of T-periodic disturbances ($T = 2\pi$), which can be generated by the system (1) with (10). Assumption 1 is satisfied. If A_{w_0}, A_{w_1}, and A_{w_2} are chosen as in (12), then Ω in (21) becomes

$$\Omega = \begin{bmatrix} (-2 + \mu)I_3 & (1-\alpha)I_3 \\ (1-\alpha)I_3 & -\mu I_3 \end{bmatrix}.$$

If $\alpha = 0$, then $\sigma_0 = 0$. The inequality (21) is satisfied with $\mu = 1$. If $\alpha > 0$, then $\sigma_0 > 0$. The inequality (21) is satisfied with $\mu = 1$ and $\varepsilon = \frac{\alpha}{2\sigma_0}$. The IMBC (16) can be written as

$$\begin{aligned}\alpha\dot{\hat{x}}_w(t) =& -\hat{x}_w(t) + (1-\alpha)\hat{x}_w(t-T) + 2x(t)\\ \hat{w}(t) =& \hat{x}_w(t), \hat{x}_w(\theta) = 0, \theta \in [-T, 0].\end{aligned} \tag{38}$$

DOI: 10.1002/acs

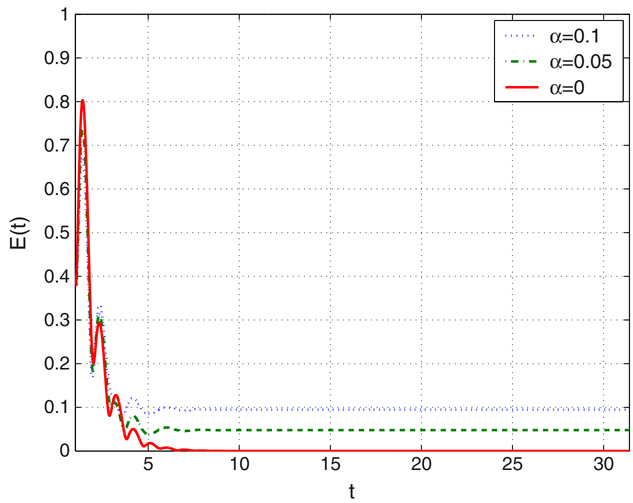

Figure 3. The evolution of $E(t)$ with different α under step disturbances.

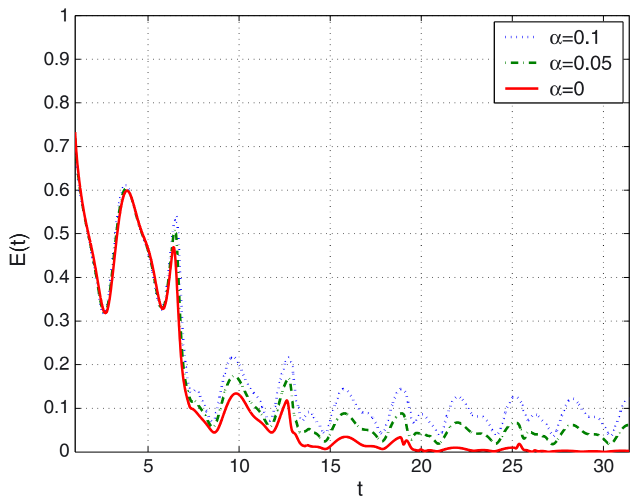

Figure 4. The evolution of $E(t)$ with different α under T-periodic disturbances.

Under Assumptions 1 and 2, we can conclude by Theorem 1 that if $\alpha = 0$, then $e \in \mathcal{L}_\infty[0,\infty) \cap \mathcal{L}_2[0,\infty)$. If $\alpha > 0$, then the tracking error $e(t)$ is uniformly ultimately bounded. As seen in Figure 4, $E(t)$ approaches 0 as t increases with $\alpha = 0$, and $E(t)$ is bounded with $\alpha = 0.1$ or 0.05. These results are consistent with conclusion (i) and conclusion (ii) in Theorem 1, respectively.

Remark 3

Take the controller (38) for example. The Laplace transform of (38) is

$$\begin{aligned} \alpha s \hat{x}_w(s) &= -\hat{x}_w(s) + (1-\alpha)\hat{x}_w(s)\exp(-sT) + 2x(s) \\ \hat{w}(s) &= \hat{x}_w(s) \end{aligned}$$

where $\hat{x}_w(s)$, $\hat{w}(s)$ and $x(s)$ are the Laplace transforms of $\hat{x}_w(t)$, $\hat{w}(t)$, and $x(t)$, respectively. Then we have

$$\hat{w}(s) = \frac{1}{1 - Q(s)\exp(-sT)} I_3 \cdot \frac{1}{1-\alpha} Q(s)x(s)$$

DOI: 10.1002/acs

where $Q(s) = \frac{1-\alpha}{\alpha s+1}$ is a filter. Therefore, the IMBC (38) can recover a filtered repetitive controller. In particular, with $\alpha = 0$, the IMBC (38) reduces to

$$\begin{aligned} \hat{x}_w(t) =& \hat{x}_w(t-T) + 2x(t) \\ \hat{w}(t) =& \hat{x}_w(t), \hat{x}_w(\theta) = 0, \theta \in [-T, 0] \end{aligned}$$

which is a repetitive controller. As shown in Figures 3 and 4, the rejection performance improves as α decreases. This is consistent with the conclusion for linear systems that as the bandwidth of $Q(s) = \frac{1-\alpha}{\alpha s+1}$ increases, that is, α decreases, the rejection performance improves, and vice versa [11, 12]. On the other hand, when $\alpha = 0$, we only have $e \in \mathcal{L}_\infty[0,\infty) \cap \mathcal{L}_2[0,\infty)$. The closed-loop error dynamics are nonexponentially stable. Whereas, when $\alpha > 0$, the closed-loop error dynamics (18) without $\sigma(t)$, namely $E\dot{z}(t) = f_a(t, z(t)) + f_d(z(t-T))$, are in fact exponentially stable. This implies that the latter is more stable than the former. Therefore, the designed IMBC can provide the flexibility to choose its parameters to achieve a tradeoff between rejection performance and stability as well.

6. CONCLUSIONS

For a class of nonlinear systems, a time-domain method is developed to design an IMBC to reject an external signal generated by a class of infinite-dimensional systems. This kind of systems can represent not only a finite-dimensional system but also an infinite-dimensional system. Therefore, by using the proposed design method, the IMBC can be designed with flexibility depending on the type of the external signal. Furthermore, by a proposed description of the external signal, the designed IMBC is flexible to choose its parameters to achieve a tradeoff between rejection performance and stability.

ACKNOWLEDGEMENTS

This work was supported by the National Natural Science Foundation of China (61104012), the 973 Program (2010CB327904), and Specialized Research Fund for the Doctoral Program of Higher Education (20111102120008).

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DOI: 10.1002/acs