# STABILITY ANALYSIS OF A CLASS OF NEUTRAL TYPE SYSTEMS IN A CRITICAL CASE WITHOUT RESTRICTION ON THE PRINCIPAL NEUTRAL TERM

Quan Quan and Kai-Yuan Cai

## ABSTRACT

This study mainly focuses on the stability of a class of linear neutral systems in a critical case, that is, the spectral radius of the principal neutral term (matrix H in this paper) is equal to 1. It is difficult to determine the stability of such systems by using existing methods. In this study, a sufficient stability criterion for the critical case without restrictions on the principal neutral term is given in terms of the existence of solutions to a linear matrix inequality.

*Key Words:* Critical case; Neutral type systems; Model transformation.

## I. Introduction

The principal neutral term usually plays an important role in the stability analysis of a neutral system. For clarity, we first introduce a class of linear neutral systems

$$\dot{x}(t) - H\dot{x}(t-\tau) = F(x_t) \qquad (1)$$

where $\tau > 0$ is a constant delay, $F(\cdot)$ is a linear functional and $x_t \triangleq x(t+\theta)$, $\theta \in [-\tau, 0]$. According to the spectral radius of matrix $H$, the neutral system (1) can be classified into three cases: $\rho(H) < 1$, $\rho(H) > 1$ and $\rho(H) = 1$. The case $\rho(H) < 1$, namely matrix $H$ is Schur stable, is a necessary condition for exponential stability of the linear neutral system (1) [1, p. 30, Corollary 7.1],[2]. To the best knowledge of the authors, the case $\rho(H) > 1$ usually leads the linear neutral system (1) to instability, for there exist characteristic roots with positive real parts [1, p. 29, Lemma 7.1]. The

last case $\rho(H) = 1$ is the critical case which is mainly concerned in this paper.

Neutral systems in the critical case need to be considered in practice because they are in fact related to a class of repetitive control systems [3],[4]. However, it is difficult to determine the stability of such systems because their characteristic equations may have an infinite sequence of roots with negative real parts approaching zero. In recent years, stability problem of neutral systems in the critical case is investigated by frequency-domain methods [5],[6] (the interested readers could consult [5] and references therein for the development on such a problem). Unfortunately, the frequency-domain stability criteria are becoming increasingly difficult as the dimension of matrix $H$ increases. As pointed out in [7], the difficulty remains as well when time-domain methods are used. To the best of the authors' knowledge, the existing stability theorems of the direct Lyapunov's method for neutral type systems cannot be applicable to the critical case.

Taking this into account, Quan et al. in [7] first proposed a linear matrix inequality (LMI) approach to analyze the stability of a class of linear neutral systems in the critical case. However, [7] requires the principal neutral term, namely matrix $H$, to be nonsingular. In this paper, a new stability criterion is proposed to remove the restriction on the principal neutral term by using a model transformation [8]. Moreover, it is proven that the model transformation is necessary to use here.

The contributions of this paper are: 1) a new criterion to analyze the stability of a class of linear neutral systems in the critical case; 2) an improvement on an existing method which broadens the applications of existing criteria.

## II. Problem Formulation and Model Transformation

Consider a class of neutral type systems

$$0 = F(A_0, A_1, x, \dot{x}) \tag{2}$$

with the initial condition

$$x(\theta) = \phi(\theta), \ \theta \in [t_0 - \tau, t_0],$$

where $F(X_0, X_1, x, v) \triangleq v(t) - Hv(t - \tau) - X_0 x(t) - X_1 x(t - \tau)$, $x(t) \in \mathbb{R}^n$, $\tau > 0$ is a constant delay and $H, A_0, A_1 \in \mathbb{R}^{n \times n}$ are constant system matrices. $\phi(t)$ is a continuously differentiable smooth vector valued function representing the initial condition function for the interval of $[t_0 - \tau, t_0]$. Suppose $x(t_0, \phi)(t)$ to be the solution of (2). For simplicity, let $t_0 = 0$ and denote $x(t_0, \phi)(t)$ to be $x(t, \phi)$ here. It is proven in [1, pp. 26–27, Theorem 7.1] that the solution $x(t, \phi)$ is unique and, continuously differentiable except maybe at the points $k\tau$, $k = 0, 1, 2 \cdots$. The purpose of this paper is to derive a stability criterion in terms of an LMI for the linear neutral system (2) with $\rho(H) \leq 1$, especially for the critical case. Unlike [7], $H$ is not required to be nonsingular here.

Before deriving the stability criterion, we need to transform (2) by using the model transformation proposed in [8].

Define

$$y(t) \triangleq \dot{x}(t) - Sx(t) \tag{3}$$

where $S \in \mathbb{R}^{n \times n}$ is the slack matrix which needs to be designed later. Substituting $\dot{x}(t) = Sx(t) + y(t)$ into (2) yields:

$$\begin{aligned} 0 &= [Sx(t) + y(t)] - H[Sx(t - \tau) + y(t - \tau)] \\ &\quad - A_0 x(t) - A_1 x(t - \tau). \end{aligned}$$

Consequently, system (2) is transformed into the following equivalent form:

$$\begin{cases} \dot{x}(t) &= Sx(t) + y(t) \\ 0 &= F(A_0 - S, A_1 + HS, x, y) \end{cases} \tag{4}$$

where $y(t)$ can be treated as the 'fast variable' as mentioned in [9].

## III. A Stability Criterion for the Critical Case

Before proceeding further, a definition and a lemma are needed.

**Definition 1** [10, pp. 128, 157],[11]: The trivial solution of the system (2) is said to be KN-stable if for any $\varepsilon > 0$, there is a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|\phi\|_W < \delta$ implies $\|x(t_0, \phi)(t)\| < \varepsilon, t \geq t_0$. The trivial solution of the system (2) is said to be asymptotically KN-stable if the trivial solution is KN-stable, and for any $\varepsilon > 0$, there is a $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|\phi\|_W < \delta$ implies $\lim_{t \to \infty} \|x(t_0, \phi)(t)\| = 0$. The trivial solution is said to be globally asymptotically KN-stable if it is KN-stable and $\lim_{t \to \infty} \|x(t_0, \phi)(t)\| = 0$ for any initial condition $\|\phi\|_W < \infty$.

**Remark 1**: Note that the definitions of stability in [10, pp. 128, 157] are slightly different from these proposed in [1, p. 130]. In [1, p. 130], the initial condition is restricted by $\sup_{\theta \in [-\tau, 0]} \|\phi(\theta)\| < \delta$ rather than $\|\phi\|_W < \delta$. The latter depends on the derivative of the initial condition. To distinguish these two definitions, we say that "stable" in the sense of Kolmanovskii-Nosov is "KN-stable" here.

**Lemma 1** [7]: For any negative semidefinite matrix $\Phi = \Phi^T \in \mathbb{R}^{n \times n}$, if $\varphi_{kk} = 0$, then $\varphi_{kj} = 0$ and $\varphi_{jk} = 0$, $j = 1, \cdots, n$, where $\varphi_{ij}$ corresponds to the element in the $i$th row and $j$th column of $\Phi$.

With *Definition 1* and *Lemma 1*, we can state the following theorem.

**Theorem 1**. If there exist $S \in \mathbb{R}^{n \times n}$, $0 < P_1 = P_1^T, P_2, P_3 \in \mathbb{R}^{n \times n}$ and $0 < Q_1 = Q_1^T \in \mathbb{R}^{n \times n}$, $0 \leq Q_2 = Q_2^T \in \mathbb{R}^{n \times n}$, $0 < W = W^T \in \mathbb{R}^{n \times n}$ such that:

$$\Omega + LWL^T \leq 0 \tag{5}$$

then the solution $x(t, \phi)$ of neutral type system (2) is globally asymptotically KN-stable, where

$$L = \begin{bmatrix} I_n & 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \end{bmatrix}^T,$$

$$\Omega = \begin{bmatrix} M_{11} & M_{12} & P_2^T(A_1 + HS) & P_2^T H \\ * & M_{22} & P_3^T(A_1 + HS) & P_3^T H \\ * & * & -Q_2 & 0_{n \times n} \\ * & * & * & -Q_1 \end{bmatrix}$$

$$M_{11} = P_1 S + S^T P_1 + (A_0 - S)^T P_2 + P_2^T(A_0 - S) + Q_2$$

$$M_{12} = P_1 - P_2^T + (A_0 - S)^T P_3$$

$$M_{22} = -P_3^T - P_3 + Q_1.$$

**Proof**. The Lyapunov functional is chosen to be

$$V(t) = x(t)^T P_1 x(t) + \int_{t-\tau}^t y(s)^T Q_1 y(s)\, ds$$
$$+ \int_{t-\tau}^t x(s)^T Q_2 x(s)\, ds. \qquad (6)$$

The time derivative of $V(t)$ is

$$\dot{V}(t) = 2x(t)^T P_1 \dot{x}(t) + y(t)^T Q_1 y(t)$$
$$- y(t-\tau)^T Q_1 y(t-\tau)$$
$$+ x(t)^T Q_2 x(t) - x(t-\tau)^T Q_2 x(t-\tau).$$

Introducing a zero term

$$[P_2 x(t) + P_3 y(t)]^T F_1 (A_0 - S, A_1 + HS, x, y, t) = 0$$

into equation above yields

$$\dot{V}(t) = \xi^T(t)\,\Omega\,\xi(t) \qquad (7)$$

where

$$\xi(t) \triangleq \left[\begin{array}{cccc} x(t)^T & y(t)^T & x(t-\tau)^T & y(t-\tau)^T \end{array}\right]^T.$$

Since $\Omega \le -LWL^T$ by (5), equation (7) becomes

$$\dot{V}(t) \le -\xi(t)^T LWL^T \xi(t)$$
$$= -x(t)^T W x(t). \qquad (8)$$

Up to now, we can conclude this proof for the case where $\rho(H) < 1$ using the existing stability theorems. However, there is no stability theorems of the direct Lyapunov's method for the critical case as far as the authors know, which implies that we need continue to complete this proof by using some new techniques. The remainder proof is composed of four propositions: *Proposition 1* is to show $x(t,\phi)$ is bounded; *Proposition 2* is to show the solution $x(t,\phi)$ is KN-stable; *Proposition 3* is to show $x(t,\phi) \in \mathcal{L}_2([0,\infty);\mathbb{R}^n)$; *Proposition 4* is to show that $\|x(t,\phi)\|^2$ is uniformly continuous. If the four propositions are satisfied, then the solution $x(t,\phi)$ of (2) is globally asymptotically KN-stable. The outline of the proof is as follows. Let $f(t) = \int_0^t \|x(s,\phi)\|^2\, ds$, then $\dot{f}(t) = \|x(t,\phi)\|^2$. Since $\|x(t,\phi)\|^2$ is uniformly continuous by *Proposition 4*, hence $f(t)$ is a differentiable function. Moreover, $f(t)$ has a finite limit as $t \to \infty$ by *Proposition 3* and $\dot{f}(t)$ is uniformly continuous by *Propositions 4*. It follows that $\lim_{t\to\infty} x(t,\phi) = 0$ for any bounded initial condition by

*Barbalat's Lemma* [12, p. 123]. Moreover, the solution $x(t,\phi)$ is KN-stable by *Proposition 2,* therefore the solution $x(t,\phi)$ of (2) is globally asymptotically KN-stable by *Definition 1*. In *Appendix, Proposition 2* is proven in detail and the other proofs are similar to that in [7], which are omitted because of limited space. $\square$

**Remark 2**. Unlike most proofs of existing stability theorems and stability criteria, the proof of *Theorem 1* does not rely on the condition $\rho(H) < 1$. This implies that *Theorem 1* can be applicable to the critical case, and to the case $\rho(H) > 1$ as well. However, the case $\rho(H) > 1$ leads the system to instability [5]. Accordingly, this implies that we cannot find a solution to the inequality (5) when $\rho(H) > 1$. Therefore, $\rho(H)$ considered here is restricted to the case $\rho(H) \le 1$.

**Remark 3**. The front of the proof until (7) is similar to the proof of Corollary 2 in [9]. A major difference is that $y$ is used to play the role of $\dot{x}$. Compared with $\dot{x}$, $y$ has a freedom to choose the slack matrix $S$. If $S = 0$, then the front of the proof until (7) is the same as the proof of Corollary 2 in [9].

**Remark 4**. The slack matrix mentioned here is different from that proposed in [13]. The slack matrices mentioned in the later only exist in introducing zero terms, but the slack matrix mentioned here exists in the designed Lyapunov functionals not just in zero terms.

**Theorem 2**. If $H = I_n$, then a necessary condition to (5) is that $S = -A_1$.

**Proof**. We have

$$\Psi^T \Omega \Psi = \left[\begin{array}{cccc} \# & \# & \# & \# \\ * & 0_{n\times n} & Q_1(A_1+S) & \# \\ * & * & \# & \# \\ * & * & * & \# \end{array}\right]$$

when choosing $H = I_n$, where "$\#$" in matrices denotes the term which is not used in the development and

$$\Psi = \left[\begin{array}{cccc} I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & I_n & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & 0_{n\times n} \\ -(A_0-S) & I_n & -(A_1+HS) & I_n \end{array}\right].$$

Inequality (5) implies that $\Omega \le 0$, then $\Psi^T \Omega \Psi \le 0$. By *Lemma 1*, we can obtain $Q_1(A_1+S) = 0$. Note $Q_1$ is nonsingular, then inequality (5) always requires $S = -A_1$.

**Remark 5**. *Theorem 2* demonstrates the necessity of the flexibility brought by the slack matrix $S$. If the slack matrix $S$ is fixed to be $S = 0$ or $S = A_0$ as in [9] or [14], then feasible solutions usually cannot be found, unless $A_1 = 0$ or $A_1 = -A_0$, respectively. *Theorem 2* also implies that $S$ could be partly determined by a

concrete neutral type system in an usual critical case not just in the case where $H = I_n$. Therefore, the slack matrix $S$ in the transformation is important.

**Remark 6**. The LMI Control Toolbox in MAT-LAB 6.5 cannot be used to solve the optimization problem (5) directly because $\Omega + LWL^T$ is a nonlinear matrix function with respect to $S$ and $P_1, P_2, P_3$. Two modified procedures with the aid of the LMI Control Toolbox proposed in [8] can be tried to solve such an optimization problem.

**Remark 7.** The paper [15] removes the assumption on the Lipschitzian constant with respect to the delayed state derivative with a constant less than 1. However, it is clear from implications of the frequency-domain stability conditions, such as proposition 1 (ii) in [15], that the proposed results in fact relax conservatism of previous work in noncritical case. Therefore the paper [15] cannot be applied to the critical case. Compared with the previous work, the change in this paper is essential (from noncritical case to critical case).

## IV. Illustrative Examples

**Example 1**. Consider a two-dimensional system (2) with

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} -2 & -0.1 \\ 0.1 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix},$$

where $H$ is a singular matrix with $\rho(H) = 1$. For any delay $\tau > 0$, choosing

$$S = -\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix},$$

we obtain the following solution to (5) that

$$\begin{aligned} P_1 &= P_2 = \begin{bmatrix} 1 & -0.1 \\ -0.1 & 1 \end{bmatrix}, P_3 = Q_1 = I_2, \\ Q_2 &= 0_{2\times 2}, W = 0.1I_2. \end{aligned}$$

Therefore, the system in *Example 1* is globally asymptotically KN-stable. This example demonstrates the effectiveness of *Theorem 1*, which shows an improvement on the proposed criterion in [7].

**Example 2**. Consider a two-dimensional system (2) with

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_0 = \begin{bmatrix} -2 & 0.1 \\ 0.1 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} 0.4 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}$$

where $H$ is a nonsingular matrix with $\rho(H) = 1$. For any delay $\tau > 0$, choosing $S = -A_1$, we obtain the

following solution to (5) that

$$\begin{aligned} P_1 &= P_2 = \begin{bmatrix} 1.6 & 0 \\ 0 & 0.9 \end{bmatrix}, P_3 = Q_1 = I_2, \\ Q_2 &= 0_{2\times 2}, W = 0.1I_2. \end{aligned}$$

Therefore, the system in *Example 2* is globally asymptotically KN-stable. This example demonstrates the effectiveness of *Theorem 2*.

## V. Conclusions

Global asymptotic stability of a class of linear neutral systems in the critical case is studied and a stability criterion in terms of an LMI is proposed. Unlike the previous work, the proposed stability criterion removes the restriction on the principal neutral term by using a model transformation. This broadens its application.

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## VI. Appendix

The following proof is similar to that in [7], so we omit some details because of limited space.

Proposition 1: $x(t, \phi)$ is bounded.

Since $W > 0$, hence $\dot{V}(t) \leq 0$ by (8). This gives $V(t) \leq V(0)$. From (6), $x(t)$ is bounded as

$$\sup_{t \in [0, \infty)} \|x(t)\| \leq b_1 \qquad (9)$$

where $b_1 = \sqrt{V(0) / \lambda_{\min}(P_1)}$. Therefore, $x(t, \phi)$ is bounded.

Proposition 2: $x(t, \phi)$ is KN-stable.

Since $\phi(t)$ is continuously differentiable, the solution $x(t, \phi)$ is continuously differentiable except maybe at the points $k\tau$, $k = 0, 1, 2 \cdots$ [1, p. 26, Theorem 7.1]. Then, by Newton–Leibniz Formula, it follows that $x(s) = x(t) - \int_s^t \dot{x}(\zeta) d\zeta$ for $s \in$

$[t - \tau, t]$. Based on the equation above, we have

$$\int_{t-\tau}^t \|x(s)\|^2 \, ds = \int_{t-\tau}^t \left\| x(t) - \int_s^t \dot{x}(\zeta) \, d\zeta \right\|^2 \, ds$$
$$\leq 2\tau \|x(t)\|^2 + 2 \int_{t-\tau}^t \left( \int_s^t \|\dot{x}(\zeta)\| \, d\zeta \right)^2 \, ds. \tag{10}$$

Using the Cauchy–Schwarz inequality $\langle a, b \rangle^2 \leq \langle a, a \rangle \langle b, b \rangle$, we obtain

$$\left( \int_s^t \|\dot{x}(\zeta)\| \, d\zeta \right)^2 \leq (t-s) \int_s^t \|\dot{x}(\zeta)\|^2 \, d\zeta$$
$$\leq \tau \int_{t-\tau}^t \|\dot{x}(\zeta)\|^2 \, d\zeta.$$

Consequently, (10) becomes

$$\int_{t-\tau}^t \|x(s)\|^2 \, ds \leq 2\tau \|x(t)\|^2 + 2\tau^2 \int_{t-\tau}^t \|\dot{x}(s)\|^2 \, ds. \tag{11}$$

Using (11), we have

$$V(t) \leq \rho_1 \|x(t)\|^2 + \rho_2 \int_{t-\tau}^t \|\dot{x}(s)\|^2 \, ds$$
$$\leq \max(\rho_1, \rho_2) \|x_t\|_W^2$$

where $\rho_1 = \lambda_{\max}(P_1) + 4\tau\lambda_{\max}(Q_1) \|S\|^2 + 2\tau\lambda_{\max}(Q_2)$ and $\rho_2 = 2\lambda_{\max}(Q_1) + 4\tau^2\lambda_{\max}(Q_1) \|S\|^2 + 2\tau^2\lambda_{\max}(Q_2)$. Therefore, $V(0) \leq \max(\rho_1, \rho_2) \|\phi\|_W^2$. For any $\varepsilon > 0$, there exists a $\delta(\varepsilon) = \varepsilon \sqrt{\lambda_{\min}(P) / \max(\rho_1, \rho_2)} > 0$ such that $\|\phi\|_W < \delta(\varepsilon)$ implies that the inequality (9) becomes

$$\sup_{t \in [0, \infty)} \|x(t)\| \leq \sqrt{\max(\rho_1, \rho_2) \|\phi\|_W^2 \Big/ \lambda_{\min}(P_1)}$$
$$= \varepsilon.$$

Therefore, the solution $x(t, \phi)$ is KN-stable by Definition 2.