

Output feedback tracking control by additive state decomposition for a class of uncertain systems

Quan Quan^{a,*}, Hai Lin^b and Kai-Yuan Cai^a

^aDepartment of Automatic Control, Beihang University (Beijing University of Aeronautics and Astronautics), Beijing 100191, China;

^bDepartment of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556, USA

(Received 22 January 2012; final version received 7 November 2012)

Besides parametric uncertainties and disturbances, unmodelled dynamics and time delay at the input are often present in practical systems, and cannot always be ignored. This paper aims to solve the problem of output feedback tracking control for a class of non-linear uncertain systems subject to unmodelled high-frequency gains and time delay in the input. By additive state decomposition, the uncertain system is transformed to an uncertainty-free system, in which the uncertainties, disturbances and effects of unmodelled dynamics along with time delay are lumped into a new disturbance at the output. Subsequently, additive state decomposition is used to decompose the transformed system to simplify the tracking controller design. The proposed control scheme is applied to three benchmark examples to demonstrate its effectiveness.

Keywords: additive state decomposition; tracking; input delay; unmodelled dynamics; output feedback; non-linear systems

1. Introduction

Tracking control of an uncertain system is a challenging problem. Most existing methods mainly focus on systems that are subject to parametric uncertainties and/or additive disturbances (see Chen and Su 2002; Ruan, Yang, Wang, and Li 2006; Nunes, Liu, and Lizarralde 2009). Studies of Corradini and Orlando (2003) and Chaturvedi et al. (2006) focused on systems that are subject to uncertainties at inputs, such as the backlash, dead zone or other nonlinearities. It is well known that unmodelled dynamics and time delay at the input are often present in practical systems, for example flight control systems (see Smith and Sarrafian 1986; Johnson, Davidson, and Murphy 1994). Such uncertainties in the input, if not properly attended, may produce a significant degradation in tracking performance or even cause instability. For example, in Rohrs, Valavani, Athans, and Stein (1985), the authors constructed a simple example, later known as Rohrs' example, to show that conventional adaptive control algorithms lose their robustness in the presence of unmodelled dynamics. Furthermore, some control algorithms, such as the repetitive control example considered in Quan and Cai (2011), may lose their robustness in the presence of input delay. Therefore, it is important to explicitly consider unmodelled dynamics and time delay in the controller design.

In this paper, the output feedback tracking control problem is investigated for a class of single-input single-output (SISO) non-linear systems subject to mismatching parametric uncertainties, mismatching additive disturbances, unmodelled high-frequency gains and time delay at the

input. This control problem has attracted a lot of attention, and many methods have been developed in the literature. A direct way is to estimate all the unknown parameters, then compensate for them. In Bresch-Pietri and Krstic (2009), a tracking problem for a linear system subject to unknown parameters and an unknown input delay was considered, where both the parameters and input delay were estimated by the proposed method. However, this method cannot handle non-parametric uncertainties such as unmodelled high-frequency gains. The second way is to design an adaptive controller to compensate for a set of unknown parameters but with robustness against other uncertainties. In Xargay, Hovakimyan, and Cao (2009), the Rohrs' example and the two-cart example, which are tracking problems for uncertain linear systems subject respectively to unmodelled dynamics and time delay at the input, were revisited by $\mathcal{L}_1$ adaptive control. In Cao and Hovakimyan (2010), the authors showed that their proposed method is robust against time delay at the input. Since each unknown parameter needs an integrator to estimate (e.g. see Equations (5)–(7) in Cao and Hovakimyan (2010)) an adaptive controller may require numerous integrators for an uncertain system with many unknown parameters. This will lead to a resulting closed-loop system with a reduced stability margin. In addition, the estimates may not approach the true parameters without persistently exciting signals, which are difficult to generate in practice especially when the number of unknown parameters is large (Landau, Lozano, M'Saad, and Karimi 2011, pp. 111–118). A third way is to convert a tracking problem to a stabilisation problem by the

*Corresponding author. Email: qq_buaa@buaa.edu.cn

idea of internal model principle (see Francis and Wonham 1976), if disturbances or desired trajectories are generated by an autonomous system. In Trinh and Aldeen (1996), the problem of set point output tracking of an uncertain linear system with multiple delays in both the state and control vectors was considered. There also exist other methods to handle uncertainties. However, some of them such as high-gain feedback often cannot be applied to a practice system directly as they rely on a rapidly changing control signal to attenuate uncertainties and disturbance. The drawbacks of high-gain feedback solutions are that they may saturate the actuators or excite high-frequency modes.

Compared with the existing literature, the problem studied in this paper is more general since not only the uncertainties in input but also output feedback and mismatching are considered. For output feedback, the state needs to be estimated, which is difficult mainly due to the uncertainties and disturbances in the state equation. Even if parameters and disturbances are estimated, it is also difficult to compensate for mismatching uncertain parameters and disturbances directly. To address these difficulties, two new mechanisms are adopted in this paper. First, the input is redefined so that signal is always smooth and bounded after transmission through the unmodelled high-frequency gains and time delay at the input. And then, to handle the estimate and mismatching problem, the input-redefinition system is transformed to an uncertainty-free system, which is proven to be input–output equivalent with the aid of *additive state decomposition*.¹ All mismatching uncertainties, mismatching disturbances and effect of unmodelled dynamics along with time delays are lumped into a new disturbance at the output. An observer is then designed for the transformed system to estimate the new state and the new disturbance. Next, the transformed system is ‘additively’ decomposed into two independent subsystems in charge of the corresponding subtasks, namely the tracking (including rejection) subtask and the input-realisation subtask. Then, one can design a controller for each subtask and finally integrate them together to achieve the original control task. Three benchmark examples are given to demonstrate the effectiveness of the proposed control scheme.

Additive state decomposition is a decomposition scheme proposed in our previous work (see Quan, Cai, and Lin (2011, online)), which is used to transform output feedback tracking control for systems with measurable nonlinearities and unknown disturbances and then decompose it into three simpler problems. This makes a challenging control problem tractable. In this paper, a different control problem is investigated by using additive state decomposition. Correspondingly, the transformation and decomposition are different. The major contributions of this paper are: (i) a tracking control scheme is able to handle mismatching parametric uncertainty, mismatching additive disturbances, unmodelled high-frequency gains and time delay at the input; (ii) a model transformation is to lump various uncertainties

together; (iii) additive state decomposition simplifies the controller design, especially in handling a saturation term.

This paper is organised as follows. In Section 2, the problem formulation is given, and additive state decomposition is introduced briefly. In Section 3, the input is redefined and the input-redefinition system is transformed to an uncertainty-free system in the sense of input–output equivalence. Sequentially, the controller design is given in Section 4. In Section 5, the two-cart example is revisited by the proposed control scheme. Section 6 concludes this paper.

2. Problem formulation and additive state decomposition

2.1. Problem formulation

Consider a class of SISO non-linear systems as follows:

$$\begin{aligned}\dot{x} &= f(t, x, \theta) + bu_\xi + d, x(0) = x_0, \\ y &= c^T x.\end{aligned}\quad (1)$$

Here, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ are constant vectors, $\theta(t) \in \mathbb{R}^m$ belongs to a given compact set $\Omega \subseteq \mathbb{R}^m$, $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}$ is the output, $d(t) \in \mathbb{R}^n$ is a bounded disturbance vector and $u_\xi(t) \in \mathbb{R}$ is the control subject to an unmodelled high-frequency gain and a time delay as follows:

$$u_\xi(s) = H(s)e^{-\tau s}u(s). \quad (2)$$

Here, $H(s)$ is an unknown stable proper transfer function with $H(0) = 1$ representing the unmodelled high-frequency gain at the input and $\tau \in \mathbb{R}$ is the input delay. It is assumed that only y is available from measurement. The desired trajectory $r(t) \in \mathbb{R}$ is known a priori, $t \geq 0$. In the following, for convenience, we will drop the notation t except when necessary for clarity.

For system (1), the following assumptions are made.

Assumption 1: The function $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ satisfies $f(t, 0, \theta) \equiv 0$, and is bounded when x is bounded on $[0, \infty)$. Moreover, for given $\theta \in \Omega$, there exist positive definite matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$ such that

$$P \partial_x f(t, x, \theta) + \partial_x^T f(t, x, \theta) P \leq -Q, \forall x \in \mathbb{R}^n, \quad (3)$$

where $\partial_x f \triangleq \frac{\partial f}{\partial x} \in \mathbb{R}^{n \times n}$.

Definition 1: Cao and Hovakimyan (2008) The $\mathcal{L}_1$ gain of a stable proper SISO system is defined to be $\|G\|_{\mathcal{L}_1} = \int_0^\infty |g(t)|dt$, where $g(t)$ is the impulse response of $G(s)$.

Assumption 2: There exists a known stable proper transfer function $C(s)$ with $C(0) = 1$ such that $\|C(H - 1)\|_{\mathcal{L}_1} \leq \varepsilon_H$, $\|sC\|_{\mathcal{L}_1} \leq \varepsilon_\tau$, where $\varepsilon_H, \varepsilon_\tau \in \mathbb{R}$ are positive real.

Mean Value Theorem (Dennis and Schnabel 1983, p. 74): Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be continuously differentiable in an open convex set $D \subset \mathbb{R}^n$. For any $x, x + p \in D$, $F(x + p) - F(x) = \int_0^1 \partial_x F(x + tp) dt \cdot p$.

Under *Assumptions 1–2*, the objective here is to design a tracking controller u such that $y - r$ is ultimately bounded by a small value.

Remark 1: In practice, many controlled systems such as flight control systems are often linearised around a static equilibrium. Besides this, some non-linear systems can be linearised by feedback linearisation and backstepping technology. In these cases, the linearised system can be written as (1) with $f(t, x, \theta) = A(\theta)x$. The uncertain parameter θ , caused by system identification or modelling, is often within an acceptance bound. Therefore, the stability of $A(\theta)$ can be ensured for given $\theta \in \Omega$. For such a case, there exist positive definite matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$ such that (3) holds. Notice that we do not need to know the matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$, which are only used for analysis. The *Assumption 1* also covers a class of non-linear systems, such as (6). By the mean value theorem, we have $f(t, x, \theta) - f(t, 0, \theta) = (\int_0^1 \partial_x f(t, \mu x, \theta) d\mu)x$. Since $f(t, 0, \theta) \equiv 0$ by *Assumption 1*, $f(t, x, \theta) = (\int_0^1 \partial_x f(t, \mu x, \theta) d\mu)x$. Consequently, the system $\dot{x} = f(t, x, \theta)$ is exponentially stable by Lyapunov function $x^T P x$ and property (3). The following three benchmark systems all satisfy *Assumption 1*.

Example 1 (Rohrs' example): Consider the Rohrs' example system as follows, see Rohrs et al. (1985):

$$y(s) = \frac{2}{s+1} \frac{229}{s^2 + 30s + 229} u(s). \quad (4)$$

The nominal system is assumed to be $y(s) = \frac{2}{s+3} u(s)$ here. In this case, the system (4) can be recast in the form (1) as

$$\begin{aligned} \dot{x} &= -(3 + \theta)x + 2u_\xi, x(0) = 1, \\ y &= x, \end{aligned} \quad (5)$$

where the parameter $\theta = -2$ is assumed unknown and $H(s) = \frac{229}{s^2 + 30s + 229}$, $\tau = 0$. If $\theta \in \Omega = [-2.5, 2.5]$ and the set Ω is known, then the system (4) is stable. Consequently, *Assumption 1* is satisfied. Notice that we do not need to know the matrices P and Q . Choose $C(s) = \frac{1}{2s+1}$. Then *Assumption 2* is satisfied with $\varepsilon_H = 0.12$ and $\varepsilon_\tau = 1$.

Example 2 (Nonlinear): Consider a simple non-linear system as follows, see Hagenmeyer and Delaleau (2003):

$$\begin{aligned} \dot{x} &= -x - (1 + \theta)x^3 + u(t - \tau) + d, x(0) = 1, \\ y &= x, \end{aligned} \quad (6)$$

where $x, y, u, d \in \mathbb{R}$, the parameter $\theta(t) = 0.2 \sin(0.1t + 1)$, the input delay $\tau = 0.1$ and $d(t) = 0.5 \sin(0.2t)$ are assumed unknown. The system (6) can be recast in the form (1) with $f(t, x, \theta) = -x - (1 + \theta)x^3$ and $H(s) = 1$, $\tau = 0.1$. If $\theta \in \Omega = [-0.2, 0.2]$ and the set Ω is known, the system (6) is stable. It is easy to verify $\partial_x f(t, x, \theta) = -1 - 3(1 + \theta)x^2 \leq -1$. So, there exist $P = 1 \in \mathbb{R}$ and $Q = 2 \in \mathbb{R}$ such that (3) holds. Consequently, *Assumption 1* is satisfied. Let $C(s) = \frac{1}{2s+1}$. Then, *Assumption 2* is satisfied with $\varepsilon_H = 0$ and $\varepsilon_\tau = 1$.

Remark 2: The Rohrs' example was proposed by Charles Rohrs in 1982, see Rohrs et al. (1985), which was to demonstrate that the available adaptive control algorithms were not able to adjust the bandwidth of the closed-loop system and guarantee its robustness. The *Non-linear* example was given in Hagenmeyer and Delaleau (2003) to show robustness issues caused by using exact feedback linearisation. Readers can refer to Rohrs et al. (1985) and Hagenmeyer and Delaleau (2003) for details. The two benchmark examples imply that uncertainties either in system parameters or in the input cannot be ignored in practice when designing a tracking controller, even if the original systems are stable. This is also the initial motivation of this paper.

2.2. Additive state decomposition

In order to make the paper self-contained, additive state decomposition in Quan and Cai (2009) is recalled briefly here. Consider the following 'original' system:

$$f(t, \dot{x}, x) = 0, x(0) = x_0, \quad (7)$$

where $x \in \mathbb{R}^n$. We first bring in a 'primary' system having the same dimension as (7), according to:

$$f_p(t, \dot{x}_p, x_p) = 0, x_p(0) = x_{p,0}, \quad (8)$$

where $x_p \in \mathbb{R}^n$. From the original system (7) and the primary system (8), we derive the following 'secondary' system:

$$f(t, \dot{x}, x) - f_p(t, \dot{x}_p, x_p) = 0, x(0) = x_0, \quad (9)$$

where $x_p \in \mathbb{R}^n$ is given by the primary system (8). Define a new variable $x_s \in \mathbb{R}^n$ as follows:

$$x_s \triangleq x - x_p. \quad (10)$$

Then, the secondary system (9) can be further written as follows:

$$\begin{aligned} f(t, \dot{x}_s + \dot{x}_p, x_s + x_p) - f_p(t, \dot{x}_p, x_p) &= 0, \\ x_s(0) &= x_0 - x_{p,0}. \end{aligned} \quad (11)$$

From the definition (10), we have

$$x(t) = x_p(t) + x_s(t), t \geq 0. \quad (12)$$

Remark 3: The ‘original’ system, namely (7), denotes the system that is needed to be decomposed. Correspondingly, the ‘primary’ system, namely (8), denotes a virtual system we design. While, the ‘secondary’ system, namely (9), is determined by the ‘original’ system and the ‘primary’ system. By the additive state decomposition, the system (7) is decomposed into two subsystems with the same dimension as the original system. In this sense, our decomposition is ‘additive’. In addition, this decomposition is with respect to state. So, we call it ‘additive state decomposition’.

As a special case of (7), a class of differential dynamic systems is considered as follows:

$$\begin{aligned} \dot{x} &= f(t, x), x(0) = x_0, \\ y &= h(t, x), \end{aligned} \quad (13)$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Two systems, denoted by the primary system and (derived) secondary system, respectively, are defined as follows:

$$\begin{aligned} \dot{x}_p &= f_p(t, x_p), x_p(0) = x_{p,0}, \\ y_p &= h_p(t, x_p), \end{aligned} \quad (14)$$

and

$$\begin{aligned} \dot{x}_s &= f(t, x_p + x_s) - f_p(t, x_p), x_s(0) = x_0 - x_{p,0}, \\ y_s &= h(t, x_p + x_s) - h_p(t, x_p), \end{aligned} \quad (15)$$

where $x_s \triangleq x - x_p$ and $y_s \triangleq y - y_p$. The secondary system (15) is determined by the original system (13) and the primary system (14). From the definition, we have

$$\begin{aligned} x(t) &= x_p(t) + x_s(t), \\ y(t) &= y_p(t) + y_s(t), t \geq 0. \end{aligned} \quad (16)$$

3. Input redefinition and model transformation

Since $H(s)$ is unmodelled high-frequency gain and τ is input delay, the control signal should be smooth (low-frequency signal) so that it will maintain its original form as far as possible after transmission through $H(s)e^{-\tau s}$. Otherwise, the control signal will be distorted a lot. This explains why high-gain feedback is often avoided in practice. For such a purpose, the input is redefined to make the control signal smooth and bounded. This brings the effect of $H(s)e^{-\tau s}$ under control, i.e. the effect will be predictable and bounded.

3.1. Input redefinition

Redefine the input as follows:

$$u(s) = C(s)[\sigma_a(v)(s)],$$

where $v \in \mathbb{R}$ is the redefined control input and $\sigma_a : \mathbb{R} \rightarrow [-a, a]$ is a saturation function defined as $\sigma_a(x) \triangleq \text{sign}(x) \min(|x|, a)$. Then, u_ξ is written as

$$\begin{aligned} u_\xi(s) &= H(s)e^{-\tau s}C(s)[\sigma_a(v)(s)] \\ &= C(s)[\sigma_a(v)(s)] + \xi(s), \end{aligned} \quad (17)$$

where $\xi(s) = C(s)(H(s)e^{-\tau s} - 1)[\sigma_a(v)(s)]$ represents the effect of the unmodelled high-frequency gain and the time delay. The function $\xi(s)$ can be further written as

$$\begin{aligned} \xi(s) &= C(s)(H(s) - 1)e^{-\tau s}[\sigma_a(v)(s)] \\ &\quad + C(s)(e^{-\tau s} - 1)[\sigma_a(v)(s)]. \end{aligned} \quad (18)$$

From the definition of σ_a , we have $\sup_{-\infty < x < \infty} |\sigma_a(x)| \leq a$. In this paper, $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. By Assumption 2, ξ is bounded as follows:

$$\begin{aligned} \sup_{t \geq 0} |\xi(t)| &\leq \|C(s)(H(s) - 1)\|_{\mathcal{L}_1} a + \|sC(s)\|_{\mathcal{L}_1} \\ &\quad \times \sup_{t \geq 0} |\mathcal{L}^{-1}\{(e^{-\tau s} - 1)/s[\sigma_a(v)(s)]\}| \\ &\leq \varepsilon_H a + \varepsilon_\tau \sup_{t \geq 0} \left| \int_t^{t-\tau} \sigma_a(v(\lambda)) d\lambda \right| \\ &\leq (\varepsilon_H + \tau \varepsilon_\tau) a, \end{aligned} \quad (19)$$

where $\xi(t) = \mathcal{L}^{-1}(\xi(s))$. The input redefinition makes ξ bounded no matter what the redefined control input v is. Therefore, it can be designed freely. According to the input redefinition above, the controller (2) is rewritten as

$$u_\xi = u + \xi. \quad (20)$$

Here, $u(t) = \mathcal{L}^{-1}(C(s)[\sigma_a(v)(s)])$ can be written in the form of a state equation as follows:

$$\begin{aligned} \dot{z} &= A_z z + b_z \sigma_a(v), \\ u &= c_z^T z + d_z \sigma_a(v), \end{aligned} \quad (21)$$

where the vectors and matrices are compatibly dimensioned depending on $C(s)$. Substituting (20) into the system (1) results in

$$\begin{aligned} \dot{x} &= f(t, x, \theta) + bu + d_h, x(0) = x_0, \\ y &= c^T x, \end{aligned} \quad (22)$$

where $d_h = d + \xi$. The system (22) with the redefined controller (21) is here called the *input-redefinition system*.

3.2. Model transformation

The unknown parameter θ and the unknown disturbances d do not appear in ‘matching’ positions for the control input, i.e. θ and d do not appear as in $b(u_\xi + \theta^T x + d)$. Therefore, in a general system except for a one-dimensional system, the unknown uncertainties cannot often be compensated for directly. Even if θ and d satisfy the ‘matching condition’, it is also difficult to compensate for since the state x is unknown. To address this difficulty, we first transform the input-redefinition system (22) to an uncertainty-free system, which is proved to be input–output equivalent with the aid of the additive state decomposition as stated in *Theorem 1*. Before proving the theorem, the following lemma is needed.

Lemma 1: Consider the following system:

$$\dot{x} = f(t, x + z, \theta) - f(t, z, \theta) + \rho, \quad (23)$$

where $\rho(t) \in \mathbb{R}^n$ is bounded. Under Assumption 1, the solutions of (23) satisfy

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma \sup_{t_0 \leq s \leq t} \|\rho(s)\|, \quad (24)$$

where β is a class $\mathcal{KL}$ function² and $\gamma = 2 \frac{\lambda_{\max}^2(P)}{\lambda_{\min}(P)\lambda_{\min}(Q)}$.

Proof: By the mean value theorem, the function $f(t, x + z, \theta)$ can be written as

$$\begin{aligned} f(t, x + z, \theta) \\ = f(t, z, \theta) + \left(\int_0^1 \partial_x f(t, x + z + \mu x, \theta) d\mu \right) x. \end{aligned}$$

Then, the system (23) can be rewritten as

$$\dot{x} = \left(\int_0^1 \partial_x f(t, x + z + \mu x, \theta) d\mu \right) x + \rho. \quad (25)$$

Choose the Lyapunov function $V = x^T P x$. By Assumption 1, the derivative of V along (25) satisfies

$$\begin{aligned} \dot{V} &\leq -\lambda_{\min}(Q) \|x\|^2 + \lambda_{\max}(P) \|x\| \|\rho\| \\ &\leq -\frac{1}{2} \lambda_{\min}(Q) \|x\|^2, \quad \forall \|x\| \geq 2 \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)} \|\rho\|. \end{aligned}$$

By Theorem 4.19 in Khalil (2002, p. 176), we conclude the proof. $\square$

With Lemma 1 in hand, we have

Theorem 1: Under Assumption 1, there exists an estimate of θ , namely $\hat{\theta} \in \Omega$, such that the system (22) is input–output equivalent to the following system:

$$\begin{aligned} \dot{x}_{new} &= f(t, x_{new}, \hat{\theta}) + bu, \quad x_{new}(0) = 0, \\ y &= c^T x_{new} + d_{new}. \end{aligned} \quad (26)$$

Here x_{new} and d_{new} satisfy

$$\begin{aligned} \|x - x_{new}\| &\leq \beta(\|x_0\|, t - t_0) + \gamma \sup_{t_0 \leq s \leq t} \|d_{\hat{\theta}}(s)\|, \\ \|d_{new}\| &\leq \|c\| \beta(\|x_0\|, t - t_0) + \|c\| \gamma \sup_{t_0 \leq s \leq t} \|d_{\hat{\theta}}(s)\|, \end{aligned} \quad (27)$$

where β is a class $\mathcal{KL}$ function, $\gamma = 2 \frac{\lambda_{\max}^2(P)}{\lambda_{\min}(P)\lambda_{\min}(Q)}$ and $d_{\hat{\theta}} = f(t, x_{new}, \theta) - f(t, x_{new}, \hat{\theta}) + d_h$.

Proof: In the following, additive state decomposition is utilised to decompose system (22) first. Consider system (22) as the original system and choose the primary system as follows:

$$\begin{aligned} \dot{x}_p &= f(t, x_p, \hat{\theta}) + bu, \quad x_p(0) = 0, \\ y_p &= c^T x_p. \end{aligned} \quad (28)$$

Then, the secondary system is determined by the original system (22) and the primary system (28) with the rule (15) that

$$\begin{aligned} \dot{x}_s &= f(t, x_p + x_s, \theta) - f(t, x_p, \hat{\theta}) + d_h, \quad x_s(0) = x_0, \\ y_s &= c^T x_s. \end{aligned} \quad (29)$$

According to (16), we have $x = x_p + x_s$ and $y = y_p + y_s$. Consequently, we can get an uncertainty-free system as follows:

$$\begin{aligned} \dot{x}_p &= f(t, x_p, \hat{\theta}) + bu, \quad x_p(0) = 0, \\ y &= c^T x_p + y_s, \end{aligned}$$

where u and y are the same as in (22). Let $x_p = x_{new}$ and $d_{new} = y_s$. We can conclude that the system (22) is input–output equivalent to (26). Next, we will prove that (27) is satisfied. The system (29) can be rewritten as

$$\begin{aligned} \dot{x}_s &= f(t, x_p + x_s, \theta) - f(t, x_p, \theta) + d_{\hat{\theta}}, \quad x_s(0) = x_0, \\ y_s &= c^T x_s, \end{aligned} \quad (30)$$

where $d_{\hat{\theta}} = f(t, x_p, \theta) - f(t, x_p, \hat{\theta}) + d_h$. Then, by Lemma 1, we have

$$\begin{aligned} \|x(t) - x_{new}(t)\| &= \|x_s(t)\| \leq \beta(\|x_0\|, t - t_0) \\ &\quad + \gamma \sup_{t_0 \leq s \leq t} \|d_{\hat{\theta}}(s)\| \\ \|d_{new}(t)\| &\leq \|c\| \|x_s(t)\| \leq \|c\| \beta(\|x_0\|, t - t_0) \\ &\quad + \|c\| \gamma \sup_{t_0 \leq s \leq t} \|d_{\hat{\theta}}(s)\|. \end{aligned}$$

$\square$

For the uncertainty-free transformed system (26), we design an observer to estimate x_{new} and d_{new} , as stated in Theorem 2.

Theorem 2: Under Assumption 1, an observer is designed to estimate state x_{new} and d_{new} in (26) as follows:

$$\begin{aligned}\dot{\hat{x}}_{new} &= f(t, \hat{x}_{new}, \hat{\theta}) + bu, \hat{x}_{new}(0) = 0, \\ \hat{d}_{new} &= y - c^T \hat{x}_{new}.\end{aligned}\quad (31)$$

Then, $\hat{x}_{new} \equiv x_{new}$ and $\hat{d}_{new} \equiv d_{new}$.

Proof: By the mean value theorem, subtracting (31) from (26) results in

$$\begin{aligned}\dot{\tilde{x}}_{new} &= f(t, x_{new}, \hat{\theta}) - f(t, \hat{x}_{new}, \hat{\theta}) \\ &= \left(\int_0^1 \partial_x f(t, x_{new} + \mu \tilde{x}_{new}, \hat{\theta}) d\mu \right) \tilde{x}_{new}, \\ \tilde{x}_{new}(0) &= 0,\end{aligned}$$

where $\tilde{x}_{new} \triangleq x_{new} - \hat{x}_{new}$. Since $\tilde{x}_{new}(0) = 0$,³ so $\tilde{x}_{new} \equiv 0$. This implies that $\hat{x}_{new} \equiv x_{new}$. Consequently, by the relation $y = c^T x_{new} + d_{new}$ in (26), we have $\hat{d}_{new} \equiv d_{new}$. $\square$

Remark 4: By (21), the control signal u is always bounded. Therefore, by Lemma 1, the state x_{new} is always bounded. Consequently, by (27), d_{new} is always bounded as well. It is interesting to note that the new state x_{new} and disturbance d_{new} in the transformed system (26) can be observed directly rather than asymptotically or exponentially. This will facilitate the analysis and design later.

Example 3 (Rohrs' example, Example 1 continued): According to the input redefinition above, the Rohrs' example system (5) can be rewritten as follows:

$$\begin{aligned}\dot{x} &= -(3 + \theta)x + 2u + (d + 2\xi), \\ y &= x,\end{aligned}$$

where $\sup_{t \geq 0} |\xi(t)| \leq 0.12a$, and u is generated by $\dot{z} = -0.5z + 0.5\sigma_a(v)$, $u = z$. Then, according to (26), the uncertainty-free transformed system of (5) is

$$\begin{aligned}\dot{x}_{new} &= -(3 + \hat{\theta})x_{new} + 2u, \\ y &= x_{new} + d_{new},\end{aligned}\quad (32)$$

where $\hat{\theta}$ will be specified later.

Example 4 (Nonlinear, Example 2 continued): According to the input redefinition above, the non-linear system (6) can be rewritten as follows:

$$\begin{aligned}\dot{x} &= -x - (1 + \theta)x^3 + u + (d + \xi), \\ y &= x,\end{aligned}$$

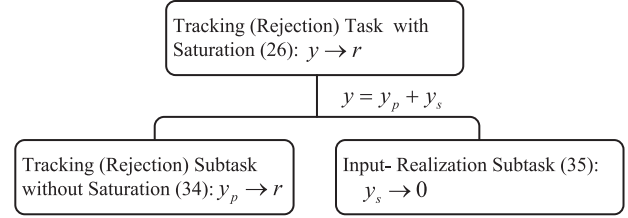


Figure 1. Additive state decomposition flow.

where $\sup_{t \geq 0} |\xi(t)| \leq 0.1a$, and u is generated by $\dot{z} = -0.5z + 0.5\sigma_a(v)$, $u = z$. Then, according to (26), the uncertainty-free transformed system of (6) is

$$\begin{aligned}\dot{x}_{new} &= -x_{new} - (1 + \hat{\theta})x_{new}^3 + u, x_{new}(0) = 0, \\ y &= x_{new} + d_{new},\end{aligned}\quad (33)$$

where $\hat{\theta}$ will be specified later.

4. Controller design

In this section, the transformed system (26) is ‘additively’ decomposed into two independent subsystems in charge of corresponding subtasks. Then, one can design a controller for each subtask, respectively, and finally combine them to achieve the original control task.

4.1. Additive state decomposition of transformed system

Currently, based on the new transformed system (26), the objective is to design a tracking controller u such that $y - r$ is ultimately bounded by a small value, while u is realised by (21). According to this fact, the transformed system (26) is ‘additively’ decomposed into two independent subsystems responsible for corresponding subtasks, namely the tracking (including rejection) subtask and the input-realisation subtask. This is shown in Figure 1.

Consider the transformed system (26) as the original system. According to the principle above, we choose the primary system as follows:

$$\begin{aligned}\dot{x}_p &= f(t, x_p, \hat{\theta}) + bu_p, x_p(0) = 0, \\ y_p &= c^T x_p + d_{new}.\end{aligned}\quad (34)$$

Then, the secondary system is determined by the original system (26) and the primary system (34) with the rule (15), and we can obtain that

$$\begin{aligned}\dot{x}_s &= f(t, x_p + x_s, \hat{\theta}) - f(t, x_p, \hat{\theta}) + b(u - u_p), \\ x_s(0) &= 0, \\ y_s &= c^T x_s.\end{aligned}\quad (35)$$

According to (16), we have

$$x_{new} = x_p + x_s \quad \text{and} \quad y = y_p + y_s. \quad (36)$$

The strategy here is to assign the tracking (including rejection) subtask to the primary system (34) and the input-realisation subtask to the secondary system (35). It is clear from (34)–(36) that if the controller u_p drives $y_p - r \rightarrow 0$ in (34) and u drives $y_s \rightarrow 0$ in (35), then $y - r \rightarrow 0$ as $t \rightarrow \infty$. The benefit brought by the additive state decomposition is that the controller u will not affect the tracking and rejection performance since the primary system (34) is independent of the secondary system (35). Since the states x_p and x_s are unknown except for addition of them, namely x_{new} , an observer is proposed to estimate x_p and x_s .

Remark 5: Although the proposed additive state decomposition makes clear how to decompose a system, it still leaves freedom to choose the primary system. By the additive state decomposition, the transformed system (26) can be also decomposed into a primary system

$$\begin{aligned} \dot{x}_p &= Ax_p + bu_p, \quad x_p(0) = 0, \\ y_p &= c^T x_p + d_{new}, \end{aligned} \quad (37)$$

and the derived secondary system

$$\begin{aligned} \dot{x}_s &= f(t, x_p + x_s, \hat{\theta}) - Ax_p + b(u_\xi - u_p), \quad x_s(0) = x_0, \\ y_s &= c^T x_s, \end{aligned} \quad (38)$$

where $A \in \mathbb{R}^{n \times n}$ is an arbitrary constant matrix. Therefore, there is an infinite number of decompositions. The principle here is to derive the secondary system with an equilibrium point as close to zero as possible. If so, the problem for the secondary system is only a stabilisation problem, which is easier compared to a tracking problem. In (35), $x_s = 0$ is an equilibrium point of $\dot{x}_s = f(t, x_p + x_s, \hat{\theta}) - f(t, x_p, \hat{\theta})$, whereas in (38), $x_s = 0$ is not an equilibrium point of $\dot{x}_s = f(t, x_p + x_s, \theta) - Ax_p$. This is why we choose the primary system as (34) not (37). As mentioned above, a good additive state decomposition often depends on the concrete problem at hand.

Theorem 3: Under Assumption 1, suppose that an observer is designed to estimate state x_p and x_s in (34)–(35) as follows:

$$\begin{aligned} \dot{\hat{x}}_p &= f(t, \hat{x}_p, \hat{\theta}) + bu_p, \quad \hat{x}_p(0) = 0, \\ \hat{x}_s &= x_{new} - \hat{x}_p. \end{aligned} \quad (39)$$

Then, $\hat{x}_p \equiv x_p$ and $\hat{x}_s \equiv x_s$.

Proof: Similar to the proof of Theorem 2. $\square$

So far, we have transformed the original system to an uncertainty-free system, in which the new state and the new

disturbance can be estimated directly. And then, decompose the transformed system into two independent subsystems in charge of corresponding subtasks. In the following, we are going to investigate the controller design with respect to the two decomposed subtasks, respectively.

4.2. Problem for tracking (including rejection) subtask

Problem 1: For (34), design a controller

$$u_p = u^r(t, x_p, r, d_{new}) \quad (40)$$

such that $y_p - r \rightarrow \mathcal{B}(\delta_r)^4$ as $t \rightarrow \infty$, meanwhile keeping the state x_p bounded, where $\delta_r \in \mathbb{R}$.

Remark 6 (on Problem 1): Since $y_p = c^T x_p + d_{new}$, Problem 1 can be also considered to design u_p such that $c^T x_p - (r - d_{new}) \rightarrow 0$. Here, the difference between r and d_{new} should be clarified. The reference r is often known a priori, i.e. $r(t + T)$ is known at time t , where $T > 0$. Moreover, its derivative is often given or can be obtained by analytic methods; whereas the new disturbance d_{new} can only be obtained at time t whose derivative can only be obtained by numerical methods. By recalling (27), the new disturbance d_{new} depends on the disturbance d , the parameters θ and $\hat{\theta}$, the effect of unmodelled high-frequency gain namely ξ , the state x_{new} , and initial value x_0 . One way of reducing the complexity is to design an observer to estimate θ , and make $\hat{\theta} \rightarrow \theta$ as $t \rightarrow \infty$. As a result, the new disturbance d_{new} finally depends on d and ξ as $t \rightarrow \infty$. In practice, a low-frequency band is often dominant in the reference signal and disturbance. Therefore, from a practical point of view, we can also modify the tracking target, namely $r - d_{new}$. For example, let $r - d_{new}$ transmit a low-pass filter to obtain its major component. If the major component of $r - d_{new}$ belongs to a fixed family of functions of time, Problem 1 can also be considered as an output regulation problem (see Isidori, Marconi, and Serrani 2003).

4.3. Problem for input-realisation subtask

As shown in Figure 1, the input-realisation subtask aims to make $y_s \rightarrow 0$. Let us investigate the secondary system (35). By Lemma 1, we have

$$\begin{aligned} \|x_s(t)\| &\leq \beta(\|x_s(t_0)\|, t - t_0) \\ &\quad + \gamma \|b\| \sup_{t_0 \leq s \leq t} \|u(s) - u_p(s)\|. \end{aligned} \quad (41)$$

This implies that $y_s \rightarrow \gamma \|b\| \|c\| \mathcal{B}(\delta_s)$ as $u - u_p \rightarrow \mathcal{B}(\delta_s)$, where $\delta_s \in \mathbb{R}$. It is noticed that u can only be realised by (21). Therefore, the problem for the input-realisation subtask can be stated as follows:

Problem 2: Given a signal u_p , design a controller $v = v^s(t, u_p)$ for (21) such that $u - u_p \rightarrow \mathcal{B}(\delta_s)$ as $t \rightarrow \infty$.

This is also a tracking problem but with a saturation constraint. Here, we give a solution to *Problem 2*. The main difficulty is how to handle the saturation in (21). Here, additive state decomposition will be used again. Taking (21) as the original system, we choose the primary system as follows:

$$\begin{aligned}\dot{z}_p &= A_z z_p + b_z v, \\ u_{zp} &= c_z^T z_p + d_z v.\end{aligned}\quad (42)$$

Then, the secondary system is determined by the original system (21) and the primary system (42) with the rule (15), and we obtain

$$\begin{aligned}\dot{z}_s &= A_z z_s + b_z (\sigma_a(v) - v), \\ u_{zs} &= c_z^T z_p + d_z (\sigma_a(v) - v).\end{aligned}\quad (43)$$

According to (16), we have $z = z_p + z_s$ and $u = u_{zp} + u_{zs}$. The benefit brought by the additive state decomposition is that the controller saturation will not affect the primary system (42). Moreover, the controller v can be designed based only on the primary system (42), where the controller v uses the state z_p not z . So, the strategy here is to design $v = v^s(t, u_p)$ in (42) to drive $u_{zp} - u_p \rightarrow 0$ as $t \rightarrow \infty$ and neglect the secondary system (43). Since $v^s(t, u_p)$ is bounded, the state of the secondary system (43) will be bounded as well. If $\sigma_a(v^s(t, u_p)) - v^s(t, u_p) \rightarrow 0$ as $t \rightarrow \infty$, then $u_{zs} \rightarrow 0$ as $t \rightarrow \infty$. Consequently, $u - u_p \rightarrow 0$ as $t \rightarrow \infty$. For (42), the transfer function from v to u_{zp} is $u_{zp}(s) = C(s)v(s)$, where

$$C(s) = c_z^T (sI - A_z)^{-1} b_z + d_z. \quad (44)$$

If $C(s)$ is designed to be minimum phase, an easy way is to design v to be

$$v(s) = C^{-1}(s)u_p(s). \quad (45)$$

The design will make the signal $\sigma_a(v)$ close to the ideal one, meanwhile maintaining the signal $\sigma_a(v)$ smooth as far as possible. By recalling (18), it will make the effect of the unmodelled high-frequency gain and the time delay ξ smaller. Once v is determined, the control signal can be generated by (21).

Remark 7: This paper focuses on a general decomposition idea rather than a detailed technology. So, we do not limit methods to solve *Problems 1–2*, which are both standard control problems. To avoid abstract, we give detailed analysis for the two problems. In addition, we give three examples to show how *Problems 1–2* are solved.

4.4. Controller integration

In summary, we have

Theorem 4: Under Assumptions 1–2, suppose (i) *Problems 1–2* are solved; (ii) the controller for system (1) (or (26)) is designed as

Observer:

$$\begin{aligned}\dot{\hat{x}}_{new} &= f(t, \hat{x}_{new}, \hat{\theta}) + bu, \hat{x}_{new}(0) = 0, \\ \dot{\hat{x}}_p &= f(t, \hat{x}_p, \hat{\theta}) + bu_p, \hat{x}_p(0) = 0, \\ \hat{d}_{new} &= y - c^T \hat{x}_{new}\end{aligned}\quad (46)$$

Controller:

$$\begin{aligned}u_p &= u^r(t, \hat{x}_p, r, \hat{d}_{new}), v = v^s(t, u_p), \\ \dot{z} &= A_z z + b_z \sigma_a(v), u = c_z^T z + d_z \sigma_a(v).\end{aligned}\quad (47)$$

Then, the output of system (1) (or (26)) satisfies $y - r \rightarrow \mathcal{B}(\delta_r + \gamma \|b\| \|c\| \delta_s)$ as $t \rightarrow \infty$, meanwhile keeping all states bounded. In particular, if $\delta_r + \delta_s = 0$, then the output in system (1) (or (26)) satisfies that $y - r \rightarrow 0$ as $t \rightarrow \infty$.

Proof: It is easy to see from the proof in *Theorems 2–3* that the observer (45) will make

$$\hat{x}_{new} \equiv x_{new}, \hat{d}_{new} \equiv d_{new}, \hat{x}_p \equiv x_p. \quad (48)$$

Suppose that *Problem 1* is solved. By (40) and (48), the controller $u_p = u^r(t, \hat{x}_p, r, \hat{d}_{new})$ can drive $y_p - r \rightarrow \mathcal{B}(\delta_r)$ as $t \rightarrow \infty$ in (34). Suppose that *Problem 2* is solved. By (48), the controller $v = v^s(t, u_p)$ can drive $u - u_p \rightarrow \mathcal{B}(\delta_s)$ as $t \rightarrow \infty$ in (35). Further by (41), we have $y_s \rightarrow \mathcal{B}(\gamma \|b\| \|c\| \delta_s)$. Since $y = y_p + y_s$, we have $y - r \rightarrow \mathcal{B}(\delta_r + \gamma \|b\| \|c\| \delta_s)$. $\square$

Example 5 (Rohrs' example, Example 3 continued): According to (34), the primary system of linear system (32) can be rewritten as follows:

$$\begin{aligned}\dot{x}_p &= -(3 + \hat{\theta})x_p + 2u_p, \\ y &= x_p + d_{new}.\end{aligned}$$

Design $u_p = \frac{1}{2}[(2 + \hat{\theta})x_p + r + \dot{r} - d_{new} - \dot{d}_{new}]$. Then, the system above becomes $\dot{e}_p = -e_p$, where $e_p = y_p - r$. Therefore, $y_p \rightarrow r$ as $t \rightarrow \infty$. According to (45), v is designed as $v^s(t, u_p) = 2\dot{u}_p + u_p$. Here, $\dot{u}_p$ and $\dot{d}_{new}$ are approximated by $\dot{d}_{new} \approx \mathcal{L}^{-1}(\frac{s}{0.1s+1}d_{new}(s))$ and $\dot{u}_p \approx \mathcal{L}^{-1}(\frac{s}{0.1s+1}u_p(s))$, respectively. Suppose $\hat{\theta} = 0$ and given $r = 0.5$ and $r = 0.5 \sin(0.2t)$, respectively. Driven by the resulting controller (47), the simulation result is shown in Figure 2.

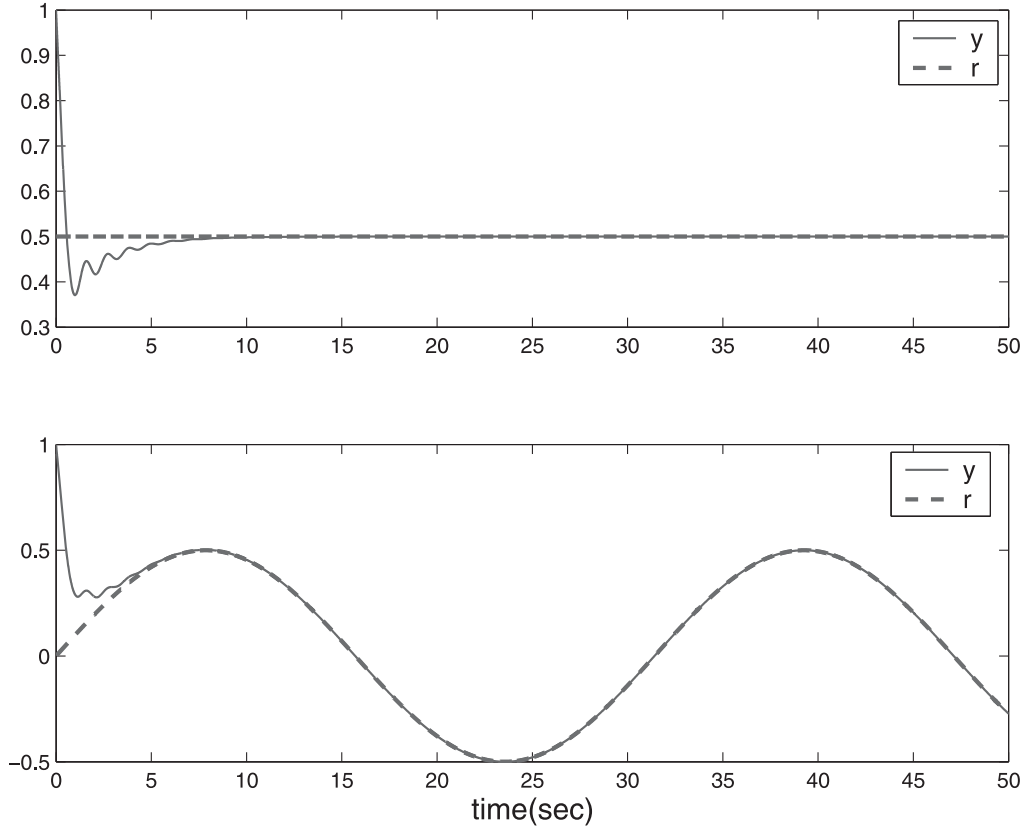


Figure 2. Output of the Rohrs' example system.

Example 6 (Nonlinear, Example 4 continued): According to (34), the primary system of non-linear system (33) can be rewritten as follows:

$$\begin{aligned}\dot{x}_p &= -x_p - (1 + \hat{\theta})x_p^3 + u_p, \quad x_p(0) = 0, \\ y_p &= x_p + d_{new}.\end{aligned}$$

Design $u_p = (1 + \hat{\theta})x_p^3 + \dot{r} + r - \dot{d}_{new} - d_{new}$. Then, the system above becomes $\dot{e}_p = -e_p$, where $e_p = y_p - r$. Therefore, $y_p - r \rightarrow 0$ as $t \rightarrow \infty$. According to (45), $v^s(t, u_p)$ is designed as $v^s(t, u_p) = 2\dot{u}_p + u_p$. Here, the derivative of u_p and d_{new} are approximated by $\dot{d}_{new} \approx \mathcal{L}^{-1}(\frac{s}{0.1s+1}d_{new}(s))$ and $\dot{u}_p \approx \mathcal{L}^{-1}(\frac{s}{0.1s+1}u_p(s))$, respectively. Suppose $\hat{\theta} = 0$ and given $r = 0.5$ and $r = 0.5 \sin(0.2t)$, respectively. Driven by the resulting controller (47), the simulation result is shown in Figure 3.

Remark 8: The derivative of d_{new} and u_p can be also obtained by the differentiator technique (see Han and Wang 1994; Levent 1998). It is interesting to note that $\hat{\theta}$ is different from θ , but $y - r$ is ultimately bounded by a small value. This is one major advantage of this proposed control scheme. Moreover, all the unknown parts such as θ , d and $H(s)e^{-s\tau}$ are treated as a lumped disturbance d_{new} . This can

explain why the proposed scheme can handle many kinds of uncertainties together.

5. Two-cart example

The two-cart mass-spring-damper example was originally proposed as a benchmark problem for robust control design (see Fekri, Athans, and Pascoal 2006; Xargay et al. 2009). Next, we will revisit the two-cart example by the proposed control scheme.

The two-cart system is shown in Figure 4. The states $x_1(t)$ and $x_2(t)$ represent the absolute positions of the two carts, whose masses are m_1 and m_2 , respectively; k_1, k_2 are the spring constants, and b_1, b_2 are the damping coefficients; $d(t)$ is a disturbance force acting on the mass m_2 ; $u(t)$ is the control force subject to an unmodelled high-frequency gain and a time delay, which acts upon the mass m_1 . The parameter $m_1 = 1$ is known, whereas the following parameters $m_2 = 2, k_1 = 0.8, k_2 = 0.5, b_1 = 1.3, b_2 = 0.9$ are assumed unknown. The unmodelled high-frequency gain and a time delay is assumed to be $H(s)e^{-\tau s} = \frac{229}{s^2 + 30s + 229}e^{-0.1s}$. The disturbance force $\zeta(t)$ is modelled as a first-order (coloured) stochastic process generated by driving a low-pass filter with continuous-time white noise $\varepsilon(s)$, with

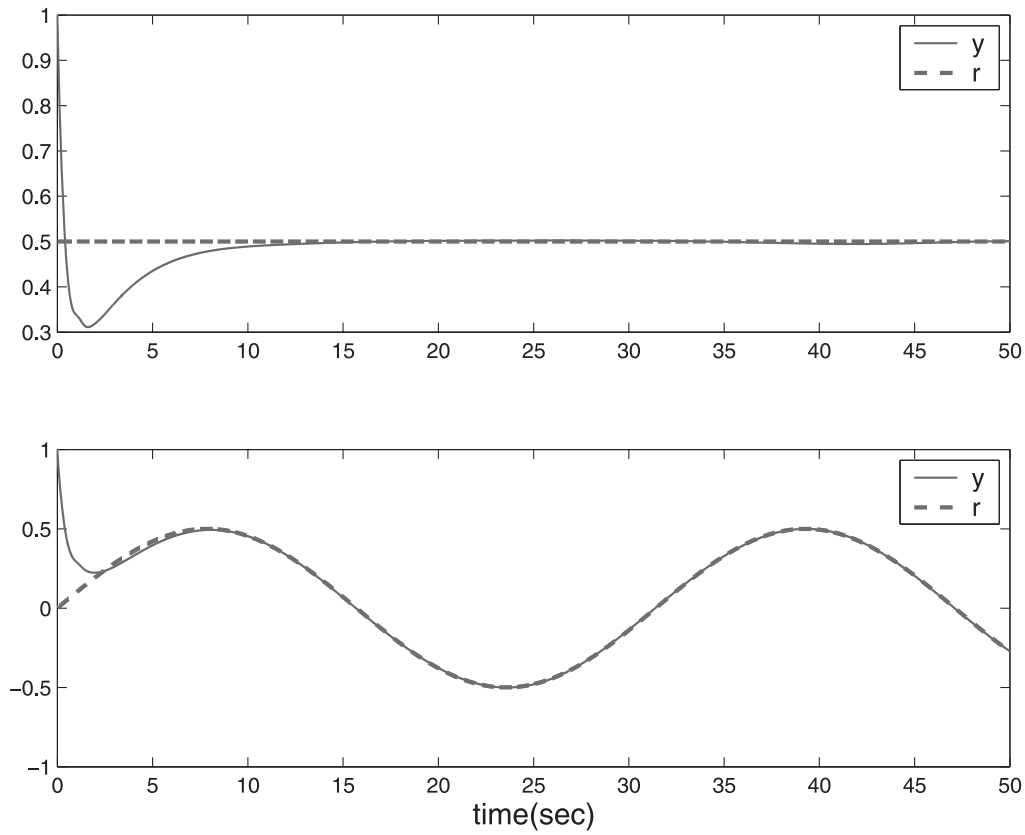


Figure 3. Output of the non-linear system.

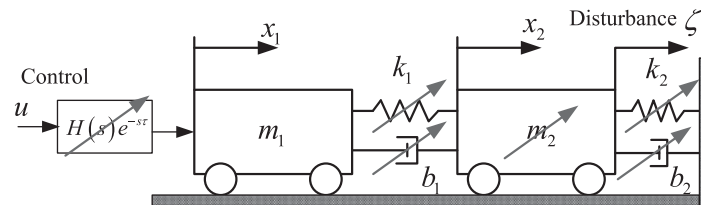


Figure 4. The two-cart system.

zero-mean and unit intensity, i.e. $\Xi = 1$, as follows form $\zeta(s) = \frac{0.1}{s+0.1} \varepsilon(s)$.

The overall state-space representation is recast in the form (1) as follows:

$$\begin{aligned} \dot{x} &= A(\theta)x + bu_{\xi} + d, \\ y &= c^T x, \end{aligned} \quad (49)$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix},$$

$$A(\theta) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & \frac{b_2}{m_2} & -\frac{b_1+b_2}{m_2} \end{bmatrix},$$

$$d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \zeta, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \frac{1}{m_1}, \quad c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \theta = \begin{bmatrix} m_1 \\ m_2 \\ k_1 \\ k_2 \\ b_1 \\ b_2 \end{bmatrix}.$$

The objective here is to design a tracking controller u such that $y - r$ is with a good tracking accuracy or is

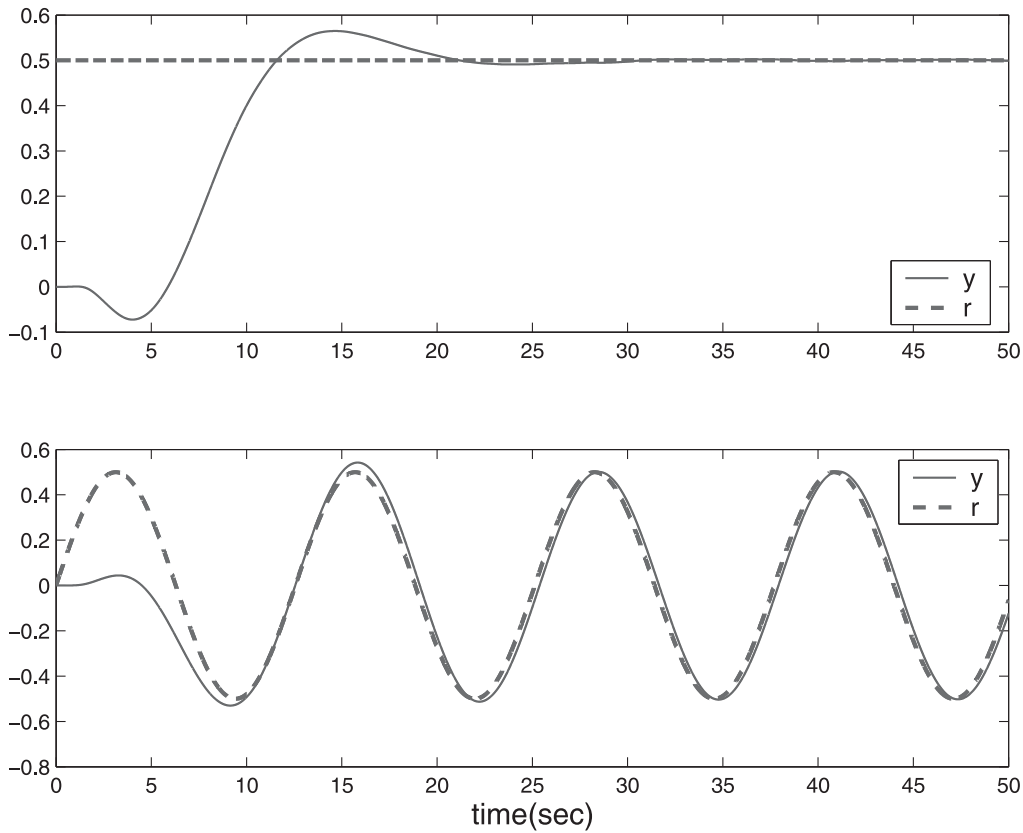


Figure 5. Output of the two-cart system in Case 1.

ultimately bounded by a small value. Since the dampers will always absorb energy until the two carts are at rest, it can be concluded that the two-cart system (a physical system) is stable. This implies that, for any given $\theta \in \Omega$, there exist positive definite matrices $P \in \mathbb{R}^{n \times n}$ and $Q \in \mathbb{R}^{n \times n}$ such that $PA(\theta) + A^T(\theta)P \leq -Q, \forall \theta \in \Omega$, where Ω represents the set that θ is normally encountered in practice. *Assumption 1* is satisfied. Let $C(s) = \frac{1}{2s+1}$. Then, *Assumption 2* is satisfied with $\varepsilon_H = 0.12$ and $\varepsilon_\tau = 1$.

According to the input redefinition above, the two-cart system (49) can be rewritten as follows:

$$\begin{aligned}\dot{x} &= A(\theta)x + bu + (d + \xi), \\ y &= c^T x,\end{aligned}$$

where $\sup_{t \geq 0} |\xi(t)| \leq 0.17a$ and u is generated by

$$\begin{aligned}\dot{z} &= -0.5z + 0.5\sigma_a(v), \\ u &= z.\end{aligned}\quad (50)$$

Then, according to (26), the uncertainty-free transformed system of (49) is

$$\begin{aligned}\dot{x}_{new} &= A(\hat{\theta})x_{new} + bu, \quad x_{new}(0) = 0, \\ y &= c^T x_{new} + d_{new},\end{aligned}\quad (51)$$

where $d_{new} = c^T e^{A(\hat{\theta})t} x_0 + \int_0^t c^T e^{A(\hat{\theta})(t-s)} b[d(s) + \xi(s) + A(\hat{\theta})x_{new}(s) - A(\hat{\theta})x_{new}(s)]ds$. According to (34), the primary system of (51) can be rewritten as follows:

$$\begin{aligned}\dot{x}_p &= A(\hat{\theta})x_p + bu_p, \quad x_p(0) = 0, \\ y_p &= c^T x_p + d_{new}.\end{aligned}\quad (52)$$

The transfer function from u_p to y_p in (52) is

$$G_{yu}(s) = c^T (sI - A(\hat{\theta}))^{-1} b,$$

which is minimum phase. Thus, u_p can be designed as

$$u_p(s) = G_{yu}^{-1}(s)(r - d_{new})(s),$$

which can drive $y_p - r \rightarrow 0$. Then, *Problem 1* is solved. Furthermore, according to (45), redefined input v in (50) is designed as $v^s(t, u_p) = \mathcal{L}^{-1}(C^{-1}(s)G_{yu}^{-1}(s)(r - d_{new})(s))$, where $C(s) = \frac{0.5}{s+0.5}$ by the definition (44).

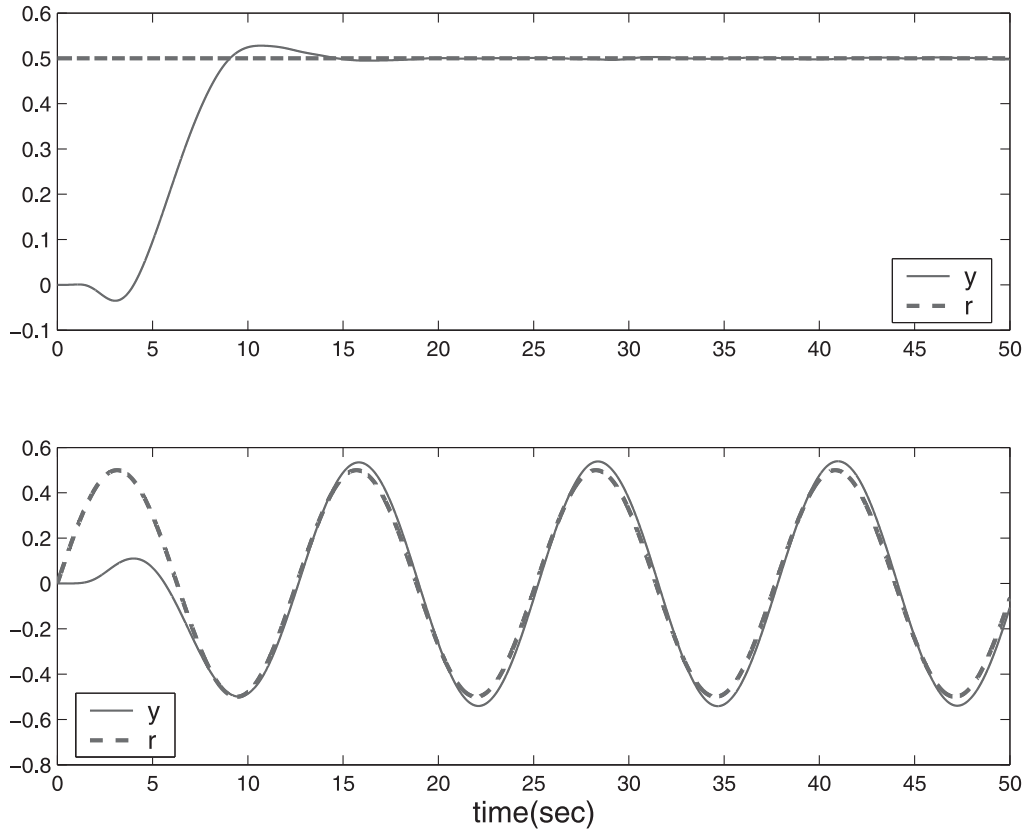


Figure 6. Output of the two-cart system in Case 2.

To realise the control, $v^s(t, u_p)$ is approximated to be

$$v^s(t, u_p) = \mathcal{L}^{-1}(Q(s)C^{-1}(s)G_{yu}^{-1}(s)(r - d_{new})(s)). \quad (53)$$

where $Q(s)$ is a fifth-order low-pass filter to make the compensator physically realisable (the order of the denominator is greater than or equal to that of numerator). For simplicity, $Q(s)$ is chosen here to be $Q(s) = \frac{1}{\prod_{k=1}^5 (\frac{1}{10k}s + 1)}$. The Problem 2 is solved. Therefore, according to (46) and (47), the controller for the two-cart system is designed as follows:

$$\begin{aligned} \dot{\hat{x}}_{new} &= A(\hat{\theta})\hat{x}_{new} + bu, \hat{x}_{new}(0) = 0, \hat{d}_{new} = y - c^T \hat{x}_{new}, \\ \dot{z} &= -0.5z + 0.5\sigma_a(v^s(t, u_p)), u = z, \end{aligned} \quad (54)$$

where $v^s(t, u_p)$ is given by (53) and here a is chosen to be 1.

To shown the effectiveness, the proposed controller (54) is applied to three cases:

Case 1: $\hat{\theta} = \theta$,

Case 2: $\hat{\theta} = [1 \ 1 \ 1 \ 0.9 \ 1.5 \ 1]^T$,

Case 3: $\theta = \hat{\theta} = [1 \ 1 \ 1 \ 0.9 \ 1.5 \ 1]^T$.

Case 1 implies that the parameters are known exactly. Case 2 implies that the parameters are unknown. While, Case 3 implies that the parameters are changed to be a specified one. The simulations are shown in Figures 5–7. The proposed controller achieves good tracking accuracy. Moreover, it is seen that the response in Cases 2–3 is faster than that in Case 1. And, the tracking accuracy in Cases 1 and 3 is better than in Case 2. So, Case 2 is a tradeoff between Case 1 and Case 3.

Remark 9: The simulations show that the proposed controller can handle the case that parameter estimates are different from the true parameters. Moreover, the response is similar to that of the model for the estimated parameters. This implies that the proposed controller, in fact, achieves results similar to those for model reference adaptive control. However, unlike model reference adaptive control, unknown parameters are not estimated and changed directly.

Remark 10: If the considered system is parameterised with many uncertain parameters, then an adaptive control often needs a corresponding number of estimators, namely integrators. This will cause the parameters to converge to true values at a very slow rate or not to converge to true values without persistent excitation. However, in the

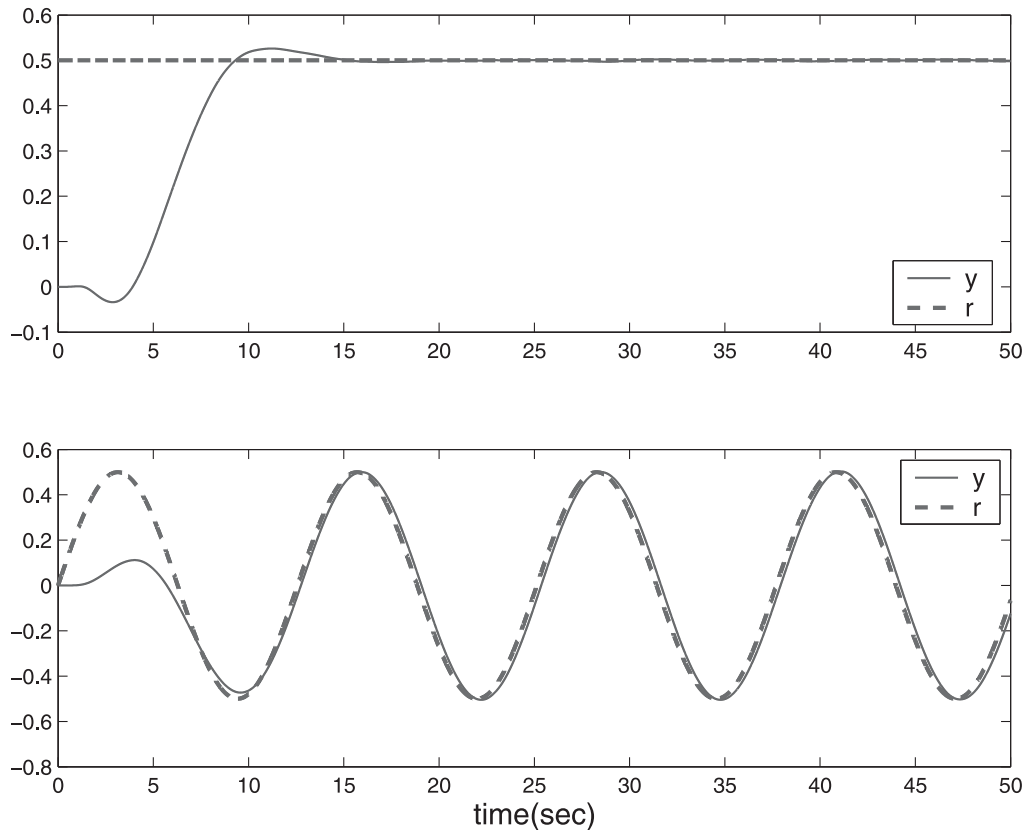


Figure 7. Output of the two-cart system in Case 3.

proposed control, five uncertain parameters and disturbances are lumped into the disturbance d_{new} , which can be estimated directly.

6. Conclusions

Output tracking control for a class of uncertain systems subject to unmodelled dynamics and time delay at the input is considered. Our main contribution lies in the presentation of a new decomposition scheme, called additive state decomposition, which not only transforms the uncertain system to an uncertainty-free system but also simplifies the controller design. The proposed control scheme has the following two salient features. Firstly, it can handle both mismatching uncertainties and disturbances. Moreover, it can achieve good tracking performance without exact parameters. Secondly, it considers many types of uncertainties together. In the presence of these uncertainties, the closed-loop system is still stable when incorporating the proposed controller. Three benchmark examples are given to show the effectiveness of the proposed control scheme.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (61104012), the 973 Program (2010CB327904)

and the Ph.D. Programs Foundation of Ministry of Education of China (20111102120008).

Notes

1. In this paper, we have replaced the term ‘additive decomposition’ in Quan and Cai (2009) with the more descriptive term ‘additive state decomposition’.
2. Refer to (Khalil 2002, p. 144) for the definition.
3. Since the initial values $x_{new}(0)$, $\hat{x}_{new}(0)$ are both assigned by the designer, they are all determinate. So, we have $\tilde{x}_{new}(0) = 0$.
4. $\mathcal{B}(\delta) \triangleq \{x \in \mathbb{R} \mid |x| \leq \delta\}$; the notation $x(t) \rightarrow \mathcal{B}(\delta)$ means $\min_{y \in \mathcal{B}(\delta)} \|x(t) - y\| \rightarrow 0$; $\mathcal{B}(\delta_1) + \mathcal{B}(\delta_2) \triangleq \{x + y \mid x \in \mathcal{B}(\delta_1), y \in \mathcal{B}(\delta_2)\}$.

Notes on contributors



Quan Quan is an Assistant Professor at the School of Automation Science and Electrical Engineering, Beihang University (Beijing University of Aeronautics and Astronautics) since 2010. He received his BS degree in 2004 and PhD degree in 2010, all from Beihang University. He was a research fellow at National University of Singapore from 7/2011– 9/2011. His research interests cover flight control and vision-based navigation and control. Web: <http://quanquan.buaa.edu.cn/>.



Hai Lin obtained his PhD degree from the University of Notre Dame in 2005. Dr Lin is currently an Assistant Professor at the Department of Electrical Engineering, University of Notre Dame. Before returning to his alma mater, Hai has been working as an Assistant Professor in the National University of Singapore from 2006 to 2011. Dr Lin's research interests are in the multidisciplinary study of the problems at the intersections of control, communication, computation and life sciences. His current research thrust is on cyber-physical systems, multi-robot cooperative tasking, and systems biology and quantum control. He is a senior member of IEEE.



Kai-Yuan Cai is a Cheung Kong Scholar (Chair Professor), jointly appointed by the Ministry of Education of China and the Li Ka Shing Foundation of Hong Kong in 1999. He has been a full Professor at Beihang University (Beijing University of Aeronautics and Astronautics) since 1995. He was born in April 1965 and entered Beihang University as an undergraduate student in 1980. He received his BS degree in 1984, MS degree in 1987 and PhD degree in 1991, all from Beihang University. He was a research fellow at City University, London, and a visiting scholar at Purdue University (USA). He was also a Visiting Professorial Fellow with the University of Wollongong, Australia. Dr Cai has published many research papers in international journals and is the author of three books: *Software Defect and Operational Profile Modeling* (Kluwer, Boston, 1998); *Introduction to Fuzzy Reliability* (Kluwer, Boston, 1996); *Elements of Software Reliability Engineering* (Tsinghua University Press, Beijing, 1995, in Chinese). He serves on the editorial board of the international journal *Fuzzy Sets and Systems*. He also served as guest editor for *Fuzzy Sets and Systems* (1996), the *International Journal of Software Engineering and Knowledge Engineering* (2006), the *Journal of Systems and Software* (2006) and *IEEE Transactions on Reliability* (2011). His main research interests include software reliability and testing, reliable flight control and software cybernetics.

References

- Bresch-Pietri, D., and Krstic, M. (2009), 'Adaptive Trajectory Tracking Despite Unknown Input Delay and Plant Parameters', *Automatica*, 45, 2074–2081.
- Cao, C., and Hovakimyan, N. (2008), 'Design and Analysis of a Novel $\mathcal{L}_1$ Adaptive Control Architecture With Guaranteed Transient Performance', *IEEE Transaction on Automatic Control*, 53, 586–591.
- Cao, C., and Hovakimyan, N. (2010), 'Stability Margins of $\mathcal{L}_1$ Adaptive Control Architecture', *IEEE Transaction on Automatic Control*, 55, 480–487.
- Chaturvedi, N.A., Sanyal, A.K., Chellappa, M., Valk, J.L., McClamroch, N.H., and Bernstein, D.S. (2006), 'Adaptive Tracking of Angular Velocity for a Planar rigid Body With Unknown Models for Inertia and Input Nonlinearity', *IEEE Transactions on Control Systems Technology*, 14, 613–627.
- Chen, X., and Su, C.-Y. (2002), 'Robust Output Tracking Control for the Systems With Uncertainties', *International Journal of Systems Science*, 33, 247–257.
- Corradini, M.L., and Orlando, G. (2003), 'Robust Practical Stabilization of Nonlinear Uncertain Plants With Input and Output Nonsmooth Nonlinearities', *IEEE Transactions on Control Systems Technology*, 11, 196–203.
- Dennis, J.E., and Schnabel, R.B. (1983), *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Englewood Cliffs, NJ: Prentice-Hall.
- Fekri, S., Athans, M., and Pascoal, A. (2006), 'Issues, Progress and New Results in Robust Adaptive Control', *International Journal of Adaptive Control and Signal Processing*, 20, 519–579.
- Francis, B.A., and Wonham, W.M. (1976), 'The Internal Model Principle of Control Theory', *Automatica*, 12, 457–465.
- Hagenmeyer, V., and Delaleau, E. (2003), 'Exact Feedforward Linearization Based on Differential Flatness', *International Journal of Control*, 76, 537–556.
- Han, J., and Wang, W. (1994), 'Nonlinear Tracking Differentiator', *Journal of Systems Science and Mathematical Sciences*, 14, 177–183.
- Isidori, A., Marconi, L., and Serrani, A. (2003), *Robust Autonomous Guidance: An Internal Model-Based Approach*, London: Springer-Verlag.
- Johnson, E.N., Davidson, J.B., and Murphy, P.C. (1994), 'Predicting the Effects of Unmodeled Dynamics on an Aircraft Flight Control System Design Using Eigenspace Assignment', *NASA Technical Memorandum*, No. 4548.
- Khalil, H.K. (2002), *Nonlinear Systems*, Upper Saddle River, NJ: Prentice-Hall.
- Landau, I.D., Lozano, R., M'Saad, M., and Karimi, A. (2011), *Adaptive Control: Algorithms, Analysis and Applications*, London: Springer.
- Levent, A. (1998), 'Robust Exact Differentiation via Sliding Mode Technique', *Automatica*, 34, 379–384.
- Nunes, E.V.L., Liu, H., and Lizarralde, F. (2009), 'Global Exact Tracking for Uncertain Systems Using Output-Feedback Sliding Mode Control', *IEEE Transactions on Automatic Control*, 54, 1141–1147.
- Quan, Q., and Cai, K.-Y. (2009), 'Additive Decomposition and Its Applications to Internal-Model-Based Tracking', in *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, China, pp. 817–822.
- Quan, Q., and Cai, K.-Y. (2011), 'A Filtered Repetitive Controller for a Class of Nonlinear Systems', *IEEE Transaction on Automatic Control*, 56, 399–405.
- Quan, Q., Cai, K.-Y., and Lin, H. (2011, online), 'Additive-Decomposition-Based Output Feedback Tracking Control for Systems With Measurable Nonlinearities and Unknown Disturbances', arXiv. <http://arxiv.org/abs/1109.4516>.
- Rohrs, C., Valavani, L., Athans, M., and Stein, G. (1985), 'Robustness of Continuous-Time Adaptive Control Algorithms in the Presence of Unmodeled Dynamics', *IEEE Transaction on Automatic Control*, 30, 881–889.
- Ruan, R.-Y., Yang, C.-L., Wang, Z.-M., and Li, Y.-Z. (2006), 'Robust Adaptive Output-Feedback Control for a Class of Nonlinear Systems With General Uncertainties', *International Journal of Systems Science*, 37, 207–224.

- Smith, R.E., and Sarrafian, S.K. (1986), 'Effect of Time Delay on Flying Qualities: An Update', *NASA Technical Memorandum*, No. 88264.
- Trinh, H., and Aldeen, M. (1996), 'Output Tracking for Linear Uncertain Time-Delay Systems', *IEEE Proceedings on Control Theory and Applications*, 143, 481–488.
- Xargay, E., Hovakimyan, N., and Cao, C. (2009), 'Benchmark Problems of Adaptive Control Revisited by $\mathcal{L}_1$ Adaptive Control', in *Proceedings of the 17th Mediterranean Conference on Control & Automation*, Makedonia Palace, Thessaloniki, Greece, pp. 31–36.