

# Repetitive Control for TORA Benchmark: An Additive-state-decomposition-based Approach

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**Abstract:** The repetitive control (RC) or repetitive controller problem for nonminimum phase nonlinear systems is both challenging and practical. In this paper, we consider an RC problem for the translational oscillator with a rotational actuator (TORA), which is a nonminimum phase nonlinear system. The major difficulty is to handle both a nonminimum phase RC problem and a nonlinear problem simultaneously. For such purpose, a new RC design, namely the additive-state-decomposition-based approach, is proposed, by which the nonminimum phase RC problem and the nonlinear problem are separated. This makes RC for the TORA benchmark tractable. To demonstrate the effectiveness of the proposed approach, a numerical simulation is given.

**Keywords:** Additive state decomposition, repetitive control, translational oscillator with a rotational actuator (TORA), rotational-translational actuator (RTAC), nonminimum phase systems.

## 1 Introduction

Repetitive control (RC) or repetitive controller is a popular topic in control theory<sup>[1–10]</sup>. By the use of frequency-domain methods, the theories and applications in linear time-invariant (LTI) systems have developed very well<sup>[1–5]</sup>. On the other hand, compared with RC in LTI systems, RC in nonlinear systems has received limited research effort<sup>[6]</sup>. Currently, there exist two major approaches to design RCs for nonlinear systems<sup>[6]</sup>. One approach, namely the linearization approach, is to transform a nonlinear system into a linear system subject to a weak nonlinear term, then apply existing frequency-domain methods to the transformed linear system. The other major approach, namely the adaptive-control-like approach<sup>[7–10]</sup>, is to convert a tracking problem for nonlinear systems into a rejection problem for nonlinear error dynamics, then apply existing adaptive-control-like design methods to the derived error dynamics. As mentioned above, the structures of RCs obtained for the linear and nonlinear systems are similar or the same, but the ways to obtain these RCs are different. For LTI systems, error dynamics are not required, whereas, for nonlinear systems, they are often required by using the adaptive-control-like approach. The requirement of error dynamics is also a key step in the general tracking design. However, this will not only restrict the development of RC but also fail to represent the special feature and importance of RC. For nonminimum phase nonlinear systems, it is difficult and computationally expensive to derive the error dynamics, es-

pecially when the internal dynamics are subject to an unknown disturbance. Because of this, we suppose that few RCs work on nonminimum phase nonlinear systems.

In this paper, we attempt to solve the rotational position tracking problem (RC problem) for a nonminimum phase nonlinear benchmark system called translational oscillator with a rotational actuator (TORA) and also known as rotational-translational actuator (RTAC)<sup>[11, 12]</sup>. It is pointed out that such RC problem is difficult to handle by both the linearization approach and the adaptive-control-like approach, because TORA is not globally feedback linearizable due to the weak, sinusoid-type nonlinear interaction between the translational oscillations and the rotational motion, and is also a nonminimum phase nonlinear system. For such purpose, we propose a new RC design under a recently developed additive-state-decomposition-based tracking control framework<sup>[13]</sup>. The key idea is to decompose the RC problem into two well-solved control problems by the additive state decomposition<sup>1</sup>: an RC for an LTI system and a state feedback stabilizing control for a nonlinear system. Since the nonminimum phase RC problem is only confined to the LTI system, existing RC methods can be applied directly and the resulting closed-loop system can be analyzed in the frequency domain. On the other hand, the state feedback stabilization is considered for the decomposed nonlinear system. This implies that the nonminimum phase behavior is avoided. This is an important feature because nonminimum phase behavior will restrict the application of basic nonlinear controllers. Fi-

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<sup>1</sup>Additive state decomposition<sup>[13]</sup> is different from the lower-order subsystem decomposition methods existing in the literature. Concretely, taking the system  $\dot{x}(t) = f(t, x)$ ,  $x \in \mathbf{R}^n$  for example, it is decomposed into two subsystems:  $\dot{x}_1(t) = f_1(t, x_1, x_2)$  and  $\dot{x}_2(t) = f_2(t, x_1, x_2)$ , where  $x_1 \in \mathbf{R}^{n_1}$  and  $x_2 \in \mathbf{R}^{n_2}$ . The lower-order subsystem decomposition satisfies  $n = n_1 + n_2$  and  $x = x_1 \oplus x_2$ . By contrast, the proposed additive state decomposition satisfies  $n = n_1 = n_2$  and  $x = x_1 + x_2$ .

nally, one can combine the RC with the stabilizing controller to achieve the original control goal. The design processes of the linearization approach, the adaptive-control-like approach and the additive-state-decomposition-based approach are different, shown in Fig. 1.

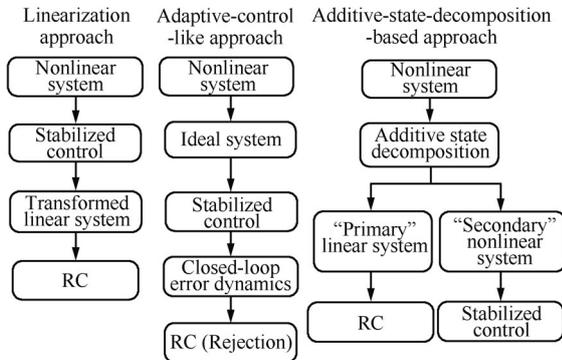


Fig. 1 Three RC design approaches

This paper focuses on an RC problem, which is different from our previous work<sup>[14]</sup>. In the previous research, the external signals are generated by finite-dimensional exosystems, whereas the considered external signals here are generated by infinite-dimensional exosystems (any periodic signal can be generated by an infinite exosystems). To make the RC system robust, we propose for the considered nonlinear system a detailed filtered RC design. As a result, uniform ultimate boundness of tracking error instead of asymptotic stability is guaranteed. Therefore, most of the design procedures and proofs here are different from those in [14].

## 2 TORA benchmark problem

The translational oscillator with a rotating actuator system or rotational-translational actuator, has been widely used as a benchmark problem to test novel nonlinear control schemes<sup>[11]</sup>. After normalization and transformation, the TORA plant is simplified by the state-space representation as<sup>[12]</sup>

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= -x_1 + \varepsilon \sin x_3 + d_2 \\
 \dot{x}_3 &= x_4 \\
 \dot{x}_4 &= \tau \\
 x(0) &= x_0
 \end{aligned} \tag{1}$$

where  $0 < \varepsilon < 1$ ,  $\tau \in \mathbf{R}$  is the dimensionless control torque and  $d_2 \in \mathbf{R}$  is an unknown disturbance.

Two assumptions are made as follows:

**Assumption 1.** The disturbance  $d_2$  is a  $T$ -periodic signal, satisfying  $d_2(t) = d_2(t - T)$ ,  $t \geq 0$ .

**Assumption 2.** All states  $x = [x_1 \ x_2 \ x_3 \ x_4]^T \in \mathbf{R}^4$  are measurable.

Under Assumptions 1 and 2, the objective is to design controller  $\tau$  such that the output  $y(t) = x_3(t) \rightarrow r(t)$  as  $t \rightarrow \infty$ , meanwhile keeping other states bounded, where

$r(t) \in (-\frac{\pi}{2}, \frac{\pi}{2})$  is a  $T$ -periodic reference and its derivative is bounded.

**Remark 1.** The TORA is not globally feedback linearizable due to the weak, sinusoid-type nonlinear interaction  $\varepsilon \sin x_3$  in (1). The internal dynamics of (1) are

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 &= -x_1.
 \end{aligned}$$

Since the internal dynamics are unstable (or say marginally stable), the considered problem is a tracking problem for a nonminimum phase nonlinear system.

**Remark 2.** In [15–19], the assumption on the disturbance is

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad d_2 = w_1$$

where  $\omega > 0$ . In such case,  $d_2$  is a special periodic signal in the form of  $\sin(\omega t + \varphi)$ , where  $\varphi \geq 0$ . By contrast, the considered disturbance in Assumption 1, generated by an infinite-dimensional exosystem  $d_2(t) = d_2(t - T)$ , is more general. The  $T$ -periodic disturbance may be generated from the vibration of other rotary mechanisms connected to the TORA plant.

## 3 Additive state decomposition of TORA benchmark

### 3.1 Additive state decomposition

In order to make the paper self-contained, the additive state decomposition<sup>[13]</sup> is recalled here briefly. Consider the following “original” system:

$$\begin{aligned}
 \dot{x} &= f(t, x) \\
 x(0) &= x_0 \\
 y &= h(t, x)
 \end{aligned} \tag{2}$$

where  $x \in \mathbf{R}^n$  and  $y \in \mathbf{R}^m$ . Two systems, denoted by the primary system and (derived) secondary system are defined as

$$\begin{aligned}
 \dot{x}_p &= f_p(t, x_p, x_s) \\
 x_p(0) &= x_{p,0} \\
 y_p &= h_p(t, x_p)
 \end{aligned} \tag{3}$$

and

$$\begin{aligned}
 \dot{x}_s &= f(t, x_p + x_s) - f_p(t, x_p, x_s) \\
 x_s(0) &= x_0 - x_{p,0} \\
 y_s &= h(t, x_p + x_s) - h_p(t, x_p)
 \end{aligned} \tag{4}$$

where  $x_s \triangleq x - x_p$  and  $y_s \triangleq y - y_p$ . The secondary system (4) is determined by the original system (2) and the primary system (3). From the definition, we have

$$x(t) = x_p(t) + x_s(t), \quad y(t) = y_p(t) + y_s(t), \quad t \geq 0. \tag{5}$$

### 3.2 Additive state decomposition of TORA benchmark

The controller  $\tau$  in (1) is designed as

$$\tau = K^T x + u$$

where  $K \in \mathbf{R}^4$  and  $u \in \mathbf{R}$  will be specified later. Then, system (1) becomes

$$\begin{aligned} \dot{x} &= (A_0 + BK^T)x + Bu + \phi(y) + d \\ y &= C^T x, \quad x(0) = x_0 \end{aligned} \tag{6}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\phi(y) = \begin{bmatrix} 0 \\ \varepsilon \sin y \\ 0 \\ 0 \end{bmatrix}, d = \begin{bmatrix} 0 \\ d_2 \\ 0 \\ 0 \end{bmatrix}.$$

By taking (6) as the ‘‘original’’ system, the primary system is chosen to be an LTI system including all external signals as

$$\begin{aligned} \dot{x}_p &= Ax_p + Bu_p + d_p \\ y_p &= C^T x_p \\ x_p(0) &= x_0 \end{aligned} \tag{7}$$

where  $A = A_0 + \varepsilon D(C + aB)^T + BK^T$ ,  $d_p = \phi(r) - \varepsilon D(r + ar) + d$  and  $D = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$ . Then, according to (4), the secondary system is derived from the original system (6) and the primary system (7) as

$$\begin{aligned} \dot{x}_s &= (A_0 + BK^T)x_s + (\phi(y_p + y_s) - \phi(r)) - \varepsilon D((C + aB)^T x_p - (r + ar)) + Bu_s \\ y_s &= C^T x_s, \quad x_s(0) = 0 \end{aligned} \tag{8}$$

where  $u_s = u - u_p$ . So, it follows

$$\begin{aligned} x &= x_p + x_s \\ y &= y_p + y_s \\ u &= u_p + u_s. \end{aligned} \tag{9}$$

As shown in Fig. 2, the TORA benchmark (6) is decomposed into two systems by the additive state decomposition: an LTI ‘‘primary’’ system (7), leaving the secondary system (8) subject to the nonlinear term  $\phi(\cdot)$ . The non-minimum phase RC task is only assigned to the LTI system (7). On the other hand, the task of handling nonlinearity is only assigned to the secondary system (8), where non-minimum phase behavior is avoided. More concretely, design an RC  $u_p$  to drive  $y_p \rightarrow r$  in (7), and design  $u_s$  to drive  $y_s$  ultimately bounded with respect to  $y_p - r$ , namely

$y_s \rightarrow 0$  as  $y_p \rightarrow r$ . If so, then  $y = y_p + y_s \rightarrow r$  by designing  $u = u_p + u_s$ . Obviously, the two tasks are easier than the original one. Therefore, the original tracking problem is simplified by the additive state decomposition.

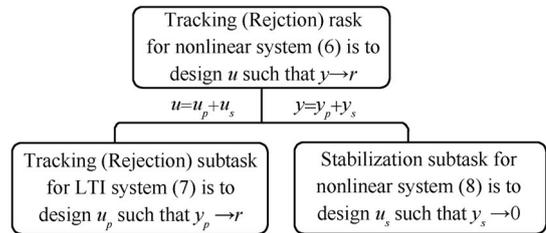


Fig. 2 Additive state decomposition of TORA

**Remark 3.** For a class of nonminimum phase systems with measurable nonlinearities and unknown disturbances, readers can find a rule to use the additive state decomposition<sup>[13]</sup>. However, besides the rule, a term  $\varepsilon D(C + aB)^T x_p - \varepsilon D(r + ar)$  is introduced additionally into (7) and (8). Note that  $(C + aB)^T x_p = y_p + ay_p$ , so the additional term  $\varepsilon D((C + aB)^T x_p - (r + ar))$  in (8) will vanish as  $y_p \rightarrow r$  and  $\dot{y}_p \rightarrow \dot{r}$ . Thanks to the additional term, the pair  $(A_0 + \varepsilon D(C + aB)^T, B)$  is controllable, whereas the pair  $(A_0, B)$  is uncontrollable. Therefore, there always exists a gain  $K$  such that the matrix  $A$  is a stable matrix.

**Remark 4.** The primary system (7) does not contain the state  $x_s$ , namely it is independent of the secondary system (8). On the other hand, the secondary system (8) depends on the output of primary system (7). If  $y_p$  has tracked  $r$ , namely  $y_p \equiv r$ , then the primary system (7) and the secondary system (8) are decoupled. According to these, the two controllers can be designed separately.

## 4 An additive-state-decomposition approach

So far, we have decomposed the transformed system into two subsystems in charge of corresponding subtasks. In this section, we are going to investigate controller design with respect to the two component subtasks respectively.

### 4.1 Filtered repetitive controller design for primary system

Define a filtered tracking error as

$$e_p = \tilde{y}_p + a\dot{\tilde{y}}_p \tag{10}$$

where  $\tilde{y}_p = y_p - r$  and  $a > 0$ . It is easy to see that both  $\tilde{y}_p$  and  $\dot{\tilde{y}}_p$  can be viewed as outputs of a stable system with  $e_p$  as input, which means that  $\tilde{y}_p$  and  $\dot{\tilde{y}}_p$  are bounded if  $e_p$  is bounded. In addition,  $\tilde{y}_p(t) \rightarrow 0$  and  $\dot{\tilde{y}}_p(t) \rightarrow 0$  as long as  $e_p(t) \rightarrow 0$ . Inspired by the design in [8], the filtered RC  $u_p$

is designed as

$$\begin{aligned} \epsilon \dot{\xi}(t) &= -\xi(t) + (1 - \epsilon^2) \xi(t - T) + L_1 e_p(t) \\ u_p(\xi, x_p, r) &= L_2^T x_p(t) + L_3 \xi(t) \\ \xi(s) &= 0, s \in [-T, 0] \end{aligned} \tag{11}$$

where  $L_1, L_3 \in \mathbf{R}$  and  $L_2 \in \mathbf{R}^4$  will be specified later. With the relation (11), one has

$$e_p(s) = \frac{1}{L_1} p(s) \xi(s)$$

where  $p(s) = \epsilon s + 1 - (1 - \epsilon^2) e^{-sT}$ . Then, the closed-loop system corresponding to the primary system (7) and (11) gives

$$\underbrace{\begin{bmatrix} \epsilon & 0_{1 \times 4} \\ 0_{4 \times 1} & I_4 \end{bmatrix}}_E \begin{bmatrix} \dot{\xi}(t) \\ \dot{x}_p(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & L_1(C + aB)^T \\ BL_3 & A + BL_2^T \end{bmatrix}}_{A_a} \begin{bmatrix} \xi(t) \\ x_p(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 - \epsilon^2 & 0_{1 \times 4} \\ 0_{4 \times 1} & 0_{4 \times 4} \end{bmatrix}}_{A_{a,-T}} \begin{bmatrix} \xi(t - T) \\ x_p(t - T) \end{bmatrix} + \underbrace{\begin{bmatrix} -L_1(r + ar)(t) \\ d_p(t) \end{bmatrix}}_{d_a(t)} \tag{12}$$

The transfer function from  $d_a$  to  $e_p$  is

$$e_p(s) = \frac{1}{L_1} p(s) \xi(s) = \frac{1}{L_1} C_a^T \underbrace{\left( sE - A_a - A_{a,-T} e^{-sT} \right)^{-1} p(s)}_{G(s)} d_a(s) \tag{13}$$

where  $C_a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ . The relation (13) can be rewritten as

$$e_p(s) = \frac{1}{L_1} \underbrace{C_a^T \left( sE - A_a - A_{a,-T} e^{-sT} \right)^{-1}}_{\text{New transfer function}} \underbrace{p(s) d_a(s)}_{\text{New input}}$$

Since  $d_a$  is a  $T$ -periodic signal,  $\mathcal{L}^{-1}(p(s) d_a(s))$  will be with a small ultimate bound, where  $\mathcal{L}^{-1}(\cdot)$  denotes the inverse Laplace transform. Especially,  $p(s) = 1 - e^{-sT}$  and  $\mathcal{L}^{-1}(p(s) d_a(s)) \rightarrow 0$  if  $\epsilon = 0$ . According to this, only the stability of  $G(s)$  (or (12) without the external signal  $d_a$ ) needs to be considered. An easy parameter choice rule is given in Theorem 1 to ensure the stable  $G(s)$ .

**Theorem 1.** Suppose that the parameters in (11) are chosen as

$$\begin{aligned} L_1 &= \kappa_1 \\ L_2 &= H - K \\ L_3 &= -\frac{2}{a} \kappa_1 \end{aligned} \tag{14}$$

where  $0 < \epsilon < 1$ ,  $\kappa_1 > 0, \kappa_2 > 2$  and  $H = -\frac{1}{a} \begin{bmatrix} 0 & \epsilon & \kappa_2 & 1 + \kappa_2 \end{bmatrix}^T$ . Then, 1) without external signal  $d_a$ , (12) is exponentially stable for any  $0 < \epsilon < 1$ ; 2) in (12),  $\|x_p\|$  and  $\|\xi\|$  are bounded on  $[0, \infty)$  and there exists  $\delta > 0$  such that<sup>2</sup>  $\|\tilde{y}_p\|_a \leq \delta$  and  $\|\dot{\tilde{y}}_p\|_a \leq \delta$ .

**Proof.** See Appendix A. □

**Remark 5.** Since the closed-loop system (12) is an LTI system, various frequency domain methods are applicable, such as the bode plot, to analyze the tracking (rejection) accuracy. This will be shown in the following numerical simulation. It should be pointed out that other RCs can be also designed for the primary system (7).

### 4.2 Stabilizing controller design for secondary system

Since  $\|\tilde{y}_p\|_a \leq \delta$  and  $\|\dot{\tilde{y}}_p\|_a \leq \delta$ , it is expected that the ultimate bound of  $\|x_s\|$  can be adjusted by  $\delta$ . For example, there exists  $\gamma > 0$ , such that  $\|x_s\|_a \leq \gamma\delta$ . If so, we have  $\|y - r\|_a \leq (1 + \gamma)\delta$ .

The secondary system (8) can be rewritten as

$$\begin{aligned} \dot{x}_{1,s} &= x_{2,s} \\ \dot{x}_{2,s} &= -x_{1,s} + \epsilon \sin(x_{3,s} + r) - \epsilon \sin r + g \\ \dot{x}_{3,s} &= x_{4,s} \\ \dot{x}_{4,s} &= K^T x_s + u_s \\ x_s(0) &= 0 \end{aligned} \tag{15}$$

where  $g = \phi(y_p + x_{3,s}) - \phi(r + x_{3,s}) - \epsilon D(C + aB)^T x_p + \epsilon D(r + ar)$ . The following constructive procedure is also given in [14]. However, in order to make the paper self-contained, the design is recalled here briefly. Since  $\|\tilde{y}_p\|_a \leq \delta$ , the proof is different from that in [14]. The controller design procedure starts from the marginally stable  $(x_{1,s}, x_{2,s})$ -subsystem.

**Step 1.** Consider the  $(x_{1,s}, x_{2,s})$ -subsystem of (16) with  $x_{3,s}$  viewed as the virtual control input. The derivative of the quadratic function  $V_1 = x_{1,s}^2 + x_{2,s}^2$  is

$$\dot{V}_1 = 2\epsilon x_{2,s} [\sin(x_{3,s} + r) - \sin r] + 2x_{2,s}g. \tag{16}$$

Then, the following ‘‘certainty equivalence’’ (CE) based virtual controller is introduced as

$$x_{3,s} = -b \arctan x_{2,s} + x'_{3,s} \tag{17}$$

where  $b \in \mathbf{R}$ . Since

$$\begin{aligned} \epsilon \sin(x_{3,s} + r) - \epsilon \sin r &= \\ \epsilon \sin(-b \arctan x_{2,s} + x'_{3,s} + r) - \epsilon \sin r &= \\ \epsilon \sin(-b \arctan x_{2,s} + r) - \epsilon \sin r + & \\ \epsilon [\sin(-b \arctan x_{2,s} + x'_{3,s} + r) - & \end{aligned}$$

<sup>2</sup>If  $x(t)$  is bounded on  $[0, \infty)$ , we let  $\|\cdot\|_a$  denote the quantity  $\|x\|_a \triangleq \lim_{t \rightarrow \infty} \sup \|x(t)\|$ .

$$\begin{aligned} & \sin(-b \arctan x_{2,s} + r) = \\ & 2\varepsilon \sin\left(\frac{-b \arctan x_{2,s}}{2}\right) \cos\left(\frac{-b \arctan x_{2,s} + 2r}{2}\right) + \\ & \varepsilon[\sin(-b \arctan x_{2,s} + x'_{3,s} + r) - \\ & \sin(-b \arctan x_{2,s} + r)] \end{aligned}$$

the  $(x_{1,s}, x_{2,s})$ -subsystem of (16) becomes

$$\begin{aligned} \dot{x}_{1,s} &= x_{2,s} \\ \dot{x}_{2,s} &= -x_{1,s} + g' + \\ & 2\varepsilon \sin\left(\frac{-b \arctan x_{2,s}}{2}\right) \cos\left(\frac{-b \arctan x_{2,s} + 2r}{2}\right) \end{aligned} \tag{18}$$

where

$$\begin{aligned} g' &= \varepsilon \sin(-b \arctan x_{2,s} + r + x'_{3,s}) - \\ & \varepsilon \sin(-b \arctan x_{2,s} + r) + g. \end{aligned} \tag{19}$$

In order to ensure  $\cos\left(\frac{-b \arctan x_{2,s} + 2r}{2}\right) > 0$ , the parameter  $b$  has to satisfy  $0 < b < \min_t 2\left(1 - \frac{2|r(t)|}{\pi}\right)$ . Since  $r(t) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ,  $b$  always exists. If  $x'_{3,s} \equiv 0$  and  $g \equiv 0$ , then  $\dot{V}_1 \leq 0$  (see Appendix B).

**Step 2.** The backstepping design is applied to the  $(x'_{3,s}, x_{4,s})$ -subsystem such that a nonlinear controller  $u_s$  drives  $x'_{3,s}$  to the origin. With the aid of (16) and (18), the time derivative of the new variable  $x'_{3,s}$  is

$$\dot{x}'_{3,s} = x_{4,s} + \psi + b \frac{1}{1 + x_{2,s}^2} g \tag{20}$$

where  $\psi = b \frac{1}{1 + x_{2,s}^2} [-x_{1,s} + \varepsilon \sin(x_{3,s} + r) - \varepsilon \sin r]$ . A new variable  $x'_{4,s}$  is defined as

$$x'_{4,s} = x'_{3,s} + x_{4,s} + \psi. \tag{21}$$

Then, (21) becomes

$$\dot{x}'_{3,s} = -x'_{3,s} + x'_{4,s} + b \frac{1}{1 + x_{2,s}^2} g.$$

By the definition (22), the time derivative of the new variable  $x'_{4,s}$  is

$$\begin{aligned} \dot{x}'_{4,s} &= \dot{x}'_{3,s} + \dot{x}_{4,s} + \dot{\psi} = \\ & -x'_{3,s} + x'_{4,s} + b \frac{1}{1 + x_{2,s}^2} g + K^T x_s + u_s + \dot{\psi}. \end{aligned}$$

Therefore,  $u_s$  for the secondary system (16) is designed as

$$u_s(x_p, x_s, r) = x'_{3,s} - 2x'_{4,s} - K^T x_s - \dot{\psi}. \tag{22}$$

Then, the  $(x'_{3,s}, x'_{4,s})$ -subsystem becomes

$$\begin{aligned} \dot{x}'_{3,s} &= -x'_{3,s} + x'_{4,s} + b \frac{1}{1 + x_{2,s}^2} g \\ \dot{x}'_{4,s} &= -x'_{4,s} + b \frac{1}{1 + x_{2,s}^2} g. \end{aligned} \tag{23}$$

We are now ready to state the theorem for the secondary system (8).

**Theorem 2.** Suppose  $\|\tilde{y}_p\|_a \leq \delta$  and  $\|\dot{\tilde{y}}_p\|_a \leq \delta$ , where  $\delta$  is sufficiently small. The controller  $u_s$  for the secondary system (16) is designed as (23), where  $0 < b < \min_t 2\left(\frac{1 - 2|r(t)|}{\pi}\right)$ . Then, there exists  $\gamma > 0$  such that  $\|x_s\|_a \leq \gamma\delta$  and  $\|x_s\|$  is bounded on  $[0, \infty)$ .

**Proof.** See Appendix B. □

**Remark 6.** As shown above, the controller design for the secondary system is somewhat complex. However, without the additive state decomposition, the controller design for the original tracking problem will be more complex. The additive state decomposition in fact makes the design more clear.

### 4.3 Controller synthesis for original system

It should be noticed that the controller design above is based on the condition that  $x_p$  and  $x_s$  are known a priori. A problem arises that the states  $x_p$  and  $x_s$  cannot be measured directly except for  $x = x_p + x_s$ . Taking this into account, the following observer is proposed to estimate the states  $x_p$  and  $x_s$ . Unlike the traditional concept of observers, the proposed observer can estimate the states of the primary system and the secondary system directly, not asymptotically or exponentially. This can be explained that, unlike a real system, either the primary system or the secondary system is in fact virtual whose initial values can be assigned freely by designers.

An observer based on the secondary system (8) can estimate  $x_s$ , which is stated in Theorem 3.

**Theorem 3.** Suppose matrix  $A$  to be stable. Then, the observer

$$\begin{aligned} \dot{\hat{x}}_s &= A\hat{x}_s + \phi(y) - \phi(r) - \\ & \varepsilon D(C + aB)^T x + \varepsilon D(r + ar) + Bu_s \\ \hat{x}_p &= x - \hat{x}_s \\ \hat{x}_s(0) &= 0 \end{aligned} \tag{24}$$

satisfies

$$\hat{x}_p \equiv x_p \text{ and } \hat{x}_s \equiv x_s.$$

**Proof.** With the aid of the relation  $x_p = x - x_s$ , the secondary system (8) can also be written as

$$\begin{aligned} \dot{x}_s &= Ax_s + \phi(y) - \phi(r) - \\ & \varepsilon D(C + aB)^T x + \varepsilon D(r + ar) + Bu_s \end{aligned} \tag{25}$$

where  $x_s(0) = 0$ . Subtracting (26) from (25) results in

$$\begin{aligned} \dot{\tilde{x}}_s &= A\tilde{x}_s \\ \tilde{x}_s(0) &= 0 \end{aligned} \tag{26}$$

where  $\tilde{x}_s = \hat{x}_s - x_s$ . Then,  $\hat{x}_p \equiv x_p$  and  $\hat{x}_s \equiv x_s$ . □

**Remark 7.** The measurement  $x$  and parameters may be inaccurate. In this case, it is expected that small uncertainties lead  $\hat{x}_p$  close to  $x_p$  (or  $\hat{x}_s$  close to  $x_s$ ). From (27), a stable matrix  $A$  can guarantee a small  $\tilde{x}_s$  in the presence of small uncertainties.

**Theorem 4.** Suppose that the conditions of Theorems 1–3 hold. If the controller  $\tau$  in system (1) is designed as

$$\begin{aligned} \epsilon \dot{\xi}(t) &= -\xi(t) + (1 - \epsilon^2) \xi(t - T) + \\ &L_1 \left( (C + aB)^T \hat{x}_p - r - ar \right) (t) \\ \tau(t) &= K^T x(t) + u_p(\xi, \hat{x}_p, r) + u_s(\hat{x}_p, \hat{x}_s, r) \end{aligned} \quad (27)$$

where  $\xi(s) = 0, s \in [-T, 0], \hat{x}_p$  and  $\hat{x}_s$  are given by (25),  $u_p(\cdot)$  is defined as in (11), and  $u_s(\cdot)$  is defined as in (23). Then, 1)  $\|x\|$  and  $\|\xi\|$  are bounded on  $[0, \infty)$  and 2) there exists  $\rho > 0$  such that  $\|y - r\|_a \leq \rho\delta$ , where  $\|\tilde{y}_p\|_a \leq \delta$  and  $\|\dot{\tilde{y}}_p\|_a \leq \delta$ .

**Proof.** Note that the original system (1), the primary system (7) and the secondary system (8) have the relation:  $x = x_p + x_s$  and  $y = y_p + y_s$ . With the controller, for the primary system (7),  $\|x_p\|, \|\xi\|$  are bounded on  $[0, \infty)$ ,  $\|\tilde{y}_p\|_a \leq \delta, \|\dot{\tilde{y}}_p\|_a \leq \delta$  by Theorem 1. On the other hand, for the secondary system (8), there exists  $\gamma > 0$  such that  $\|x_s\|_a \leq \gamma\delta$  and  $\|x_s\|$  is bounded on  $[0, \infty)$  by Theorem 2. Theorem 3 ensures that  $\hat{x}_p \equiv x_p$  and  $\hat{x}_s \equiv x_s$ . Therefore, 1)  $\|x\|$  and  $\|\xi\|$  are bounded on  $[0, \infty)$  and 2) there exists  $\rho > 0$  such that  $\|y - r\|_a \leq \rho\delta$ , where  $\rho = \gamma + 1, \|\tilde{y}_p\|_a \leq \delta$  and  $\|\dot{\tilde{y}}_p\|_a \leq \delta$ .  $\square$

The closed-loop control system is shown in Fig. 3, which includes the TORA plant (1), the controller (28) and the observer (25).

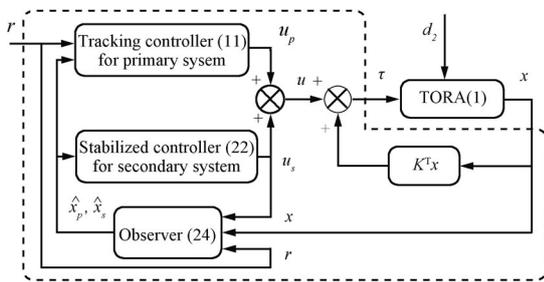


Fig. 3 Closed-loop control system of TORA

### 5 Numerical simulation

In the simulation, we set  $\epsilon = 0.2$  and the initial value  $x_0 = [0.1 \ 0 \ 0 \ 0]^T$  in (1). The reference  $r$  is chosen to be  $r(t) = 0.5 \sin(\sin \frac{2\pi}{T} t)$ , and the disturbance  $d_2(t) = 0.1 \cos(\frac{2\pi}{T} t)$ , where the period  $T = 3$ . The objective here is to design controller  $\tau$  such that the output  $y(t) - r(t)$  is uniformly ultimately bounded with a small ultimate bound. Meanwhile, other states are bounded.

The parameters are chosen as  $a = 1$  and  $K = [0 \ -\epsilon \ -1 \ -2]^T$ . Then,  $A$  in (7) satisfies

$$\max \operatorname{Re}(\lambda(A)) = -0.01 < 0.$$

Furthermore, according to Theorem 1, the parameters of the controller (28) are chosen as  $\epsilon = 0.01, L_1 = 2, L_2 = 0_{4 \times 1}$  and  $L_3 = -4$ . The parameter  $b$  is chosen as  $b = 1.5(1 - \frac{1}{\pi}) < \min_{t \in \mathbf{R}^2} (1 - \frac{2|r(t)|}{\pi})$ .

In (13), let  $G(s) = [g_1(s) \ g_2(s) \ g_3(s) \ g_4(s) \ g_5(s)]$ . With the chosen parameters above,  $|g_i(j\omega)|, i = 1, 2, \dots, 5$ , with respect to frequency  $\omega$  are plotted in Fig. 4. As shown, all  $|g_i(j\omega)|$  have comb shape with notches matching the frequencies of the considered periodic signals. This makes  $|g_i(j\omega)|$  close to zero at  $\omega = k\frac{2\pi}{T}, k = 0, 1, \dots$ , in the low frequency band. Since low frequencies dominate  $d_a(t)$ , it is expected that  $e_p$  is with a small ultimate bound.

The TORA plant (1) is driven by the controller (28) with the parameters above. The evolutions of all states of (1) are shown in Fig. 5. From the simulation, it can be seen that the proposed controller  $\tau$  drives the tracking error  $y(t) - r(t)$  to be uniformly ultimately bounded with a small ultimate bound. Meanwhile, other states are bounded. In Fig. 6, the evolutions of state and control inputs of the decomposed systems are shown. Moreover, the final control input is also plotted. As shown, the ultimate bound of  $\|x_s\|$  is small and the control  $u = u_p + u_s$  is bounded.

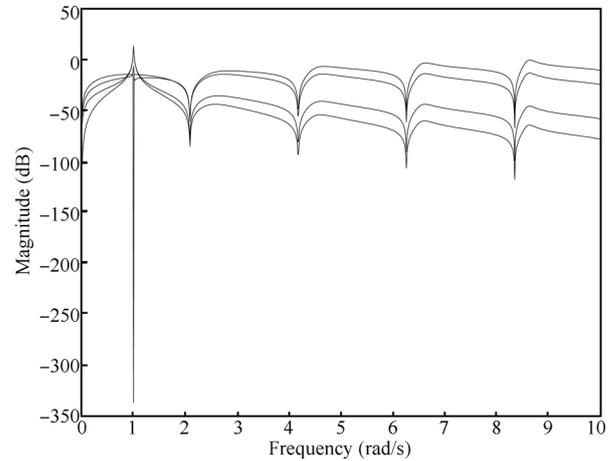


Fig. 4 Amplitude response of the transfer function of elements of  $G(s)$  in (13). In this example, the last two elements  $|g_4(j\omega)|$  and  $|g_5(j\omega)|$  are the same. Therefore, it seems that there are only four curves in the figure.

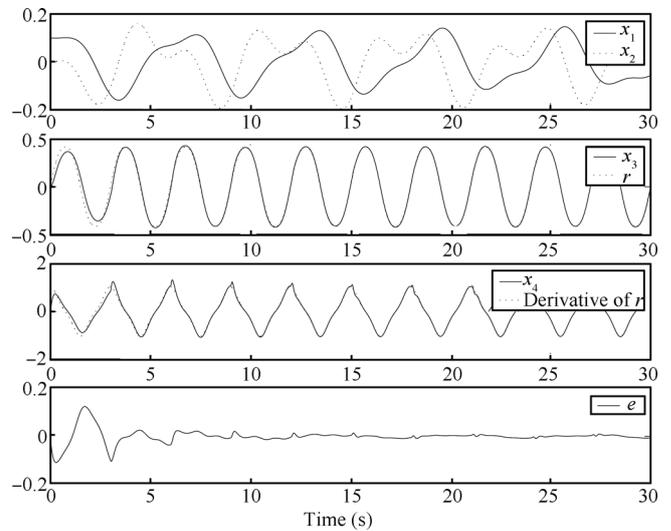


Fig. 5 Evolutions of all states

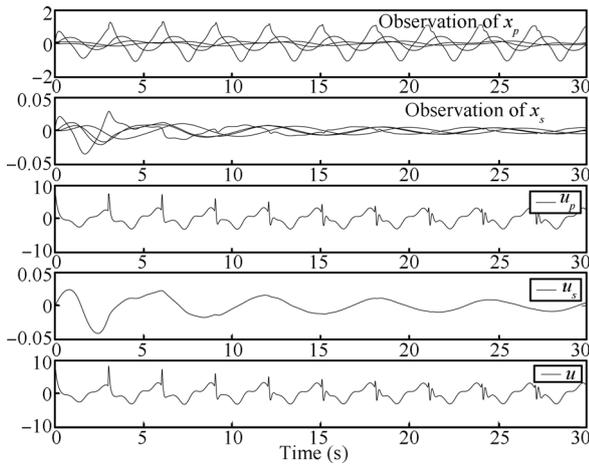


Fig. 6 Evolutions of all states and inputs of the primary system (7) and the secondary system (8)

### 6 Conclusions

In this paper, the rotational position tracking problem for TORA was solved by the proposed additive-state-decomposition-based RC. The design relied on additive state decomposition, by which the considered RC problem was decomposed into an RC problem for a “primary” LTI system and a “secondary” state-feedback stabilization problem for a nonlinear system. For the LTI system, an easy RC design is given. Thanks to its linearity, the amplitude response is applied to show the tracking accuracy Fig. 4. On the other hand, the backstepping design is applied to the state-feedback stabilization problem. This makes RC design for TORA tractable and flexible. Future works on TORA include designing RC for translational displacement tracking problem and improving the robustness of the resulting closed-loop RC system.

### Appendix

#### A. Proof of Theorem 1

Choose a Lyapunov function as

$$V = \epsilon \xi^2 + \int_{t-T}^t \xi^2(\theta) d\theta + \frac{1}{2} x_{1,p}^2 + \frac{1}{2} x_{2,p}^2 + \frac{1}{2} (x_{3,p} + a x_{4,p})^2$$

where  $x_p = [x_{1,p} \ x_{2,p} \ x_{3,p} \ x_{4,p}]^T$ . With the parameters (15), the derivative of  $V$  along (12) without external signal  $d_a$  yields

$$\dot{V} \leq -\epsilon^2 (2 - \epsilon^2) \xi^2 - \kappa_2 (x_{3,p} + a x_{4,p})^2 \leq 0.$$

Consequently, without external signal  $d_a$ , (12) is asymptotically stable for any  $0 < \epsilon < 1$ . For the linear retarded time-delay system, asymptotic stability is equivalent to exponential stability. Then, without external signal  $d_a$ , (12) is exponentially stable for any  $0 < \epsilon < 1$ . Consequently,  $\|x_p\|$  and  $\|\xi\|$  are bounded on  $[0, \infty)$  and there exists  $\delta > 0$  such that  $\|\tilde{y}_p\|_a \leq \delta$  and  $\|\dot{\tilde{y}}_p\|_a \leq \delta$ .

#### B. Proof of Theorem 2

This proof is composed of three parts.

**Part 1.** If  $\|\tilde{y}_p\|_a \leq \delta$  and  $\|\dot{\tilde{y}}_p\|_a \leq \delta$ , then from the definition of  $\phi(y)$ , we have  $\|g\|_a \leq (2 + a)\epsilon\delta$  no matter what  $y_s$  is. According to this, it is easy from (24) to see that  $\|x'_{3,s}\|_a \leq b\|g\|_a = (2 + a)b\epsilon\delta$  and  $\|x'_{4,s}\|_a = 0$  when the controller  $u_s$  for the secondary system (16) is designed as (23). Then, in (20),  $\|g'\|_a \leq \epsilon\|x'_{3,s}\|_a + \|g\|_a = (b\epsilon + 1)(2 + a)\epsilon\delta$ .

**Part 2.**  $\|x_{2,s}\| \leq c(b\epsilon + 1)(2 + a)\epsilon\delta$  and  $\|x_{2,s}\| \leq c(b\epsilon + 1)(2 + a)\epsilon\delta$ . Since  $0 < b < \min_t 2 \left(1 - \frac{2|r(t)|}{\pi}\right)$ ,  $\dot{V}_1$  in (17) is negative semidefinite when  $g'(t) \equiv 0$ , i.e.,

$$\begin{aligned} \dot{V}_1 = 2\epsilon x_{2,s} \sin\left(\frac{-b \arctan x_{2,s}}{2}\right) \times \\ \cos\left(\frac{-b \arctan x_{2,s} + 2r}{2}\right) \leq 0 \end{aligned}$$

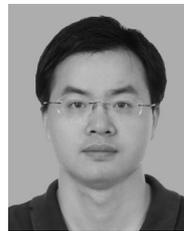
where the equality holds at some time instant  $t \geq 0$  if and only if  $x_{2,s}(t) \equiv 0$ . By LaSalle’s invariance principle<sup>[20]</sup>, it follows that  $\lim_{t \rightarrow \infty} x_{1,s}(t) = 0$  and  $\lim_{t \rightarrow \infty} x_{2,s}(t) = 0$  when  $g'(t) \equiv 0$ . Because of the particular structure of  $(x_{1,s}, x_{2,s})$ -subsystem (19), by using Lemma 3.5 in [21], it is shown that  $(x_{1,s}, x_{2,s})$ -subsystem (19) is small-input small-state with a linear gain, say  $c > 0$ . Suppose  $\delta$  is sufficiently small. Then,  $\|g'\|_a \leq (b\epsilon + 1)(2 + a)\epsilon\delta$  is sufficiently small as well. Therefore,  $\|x_{1,s}\|_a \leq c\|g'\|_a = c(b\epsilon + 1)(2 + a)\epsilon\delta$  and  $\|x_{2,s}\| \leq c(b\epsilon + 1)(2 + a)\epsilon\delta$ .

**Part 3.** From the definitions of  $x'_{3,s}$  and  $x'_{4,s}$ , we have that there exists  $\gamma > 0$  such that  $\|x_s\|_a \leq \gamma\delta$ . For the  $(x_{1,s}, x_{2,s})$ -subsystem and  $(x'_{3,s}, x'_{4,s})$ -subsystem,  $\|x_s(t)\|$  is bounded in any finite time. With the obtained result  $\|x_s\|_a \leq \gamma\delta$ , we have  $\|x_s(t)\|$  is bounded on  $[0, \infty)$ .

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