Proportional-Integral Stabilizing Control of a Class of MIMO Systems Subject to Nonparametric Uncertainties by Additive-State-Decomposition Dynamic Inversion Design

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***Abstract*—Based on the additive-state-decomposition (ASD) dynamic inversion design, a proportional-integral controller is designed to stabilize a class of multi-input multi-output (MIMO) systems subject to nonparametric time-varying uncertainties with respect to both state and input. By ASD and a new definition of output, the considered uncertain system is transformed into a first-order system, in which all original uncertainties are lumped into a new disturbance at the new defined output. Subsequently, the dynamic inversion control is applied to reject the lumped disturbance. Performance analysis of the resulting closed-loop dynamics shows that the stability can be ensured. Finally, to demonstrate its effectiveness, the proposed controller design is applied to two existing problems by numerical simulation. Furthermore, in order to show its practicability, the proposed controller design is also performed on a real quadrotor to stabilize its attitude when its inertia moment matrix is subject to a large uncertainty.**

***Index Terms*—Additive state decomposition (ASD), multiinput multioutput (MIMO) systems, PID, quadrotor, stabilization, uncertainties.**

I. Introduction

STABILIZATION in control systems with uncertainties depending on state and input has attracted the interest of many researchers. Uncertainties depending on state originate from various sources, including variations in plant parameters and inaccuracies that arise from identification. Input uncertainties include uncertain gains, dead zone nonlinearities, quantization, and backlash. In practice, these uncertainties may degrade or destabilize system performance. Therefore, robust stabilizing control problems for systems with uncertainties depending on both state and input are important.

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In this paper, a stabilizing control problem is investigated for a class of multi-input multi-output (MIMO) systems subject to nonparametric time-varying uncertainties with respect to both the state and input. Several accepted control methods to handle uncertainties are briefly reviewed. A direct approach is to estimate all unknown parameters, and then, simultaneously use such parameters to resolve uncertainties. Lyapunov methods are adopted in analyzing the stability of closed-loop systems. In [1], nonparametric uncertainties involving state and input are approximated via basis functions with unknown parameters, which are estimated by given adaptive laws. With the estimated parameters, an approximate dynamic inversion method was proposed. It is in fact an adaptive dynamic inversion method [2]. In [3], $\mathcal{L}_1$ adaptive control architecture was proposed for systems with unknown input gain, as well as unknown time-varying parameters and disturbances. The proposed control generates uniform performance bounds for a system's input and output signals, which can be systematically improved by increasing adaptation rates. In [4], an adaptive feedback controller was used to track a desired angular velocity trajectory of a planar rigid body with an unknown rotational inertia and an unknown input nonlinearity. In [5], two asymptotic tracking controllers were designed for the output tracking of an aircraft system under parametric uncertainties and unknown nonlinear disturbances, which were not linearly parameterizable. An adaptive extension was then presented, in which the feedforward adaptive estimates of input uncertainties were used. In [6], a robust adaptive feedback linearizing dynamic controller was proposed to control the system using estimated upper bounds of uncertainties. As indicated in [1]–[5], an adaptive controller may require numerous integrators that correspond to unknown parameters in an uncertain system. Each unknown parameter requires an integrator for estimation, thereby resulting in a closed-loop system with a reduced stability margin. The second direct method for resolving uncertainties is designing an inverse controller by a neural network that cancels input nonlinearities, thereby generating a linear function [7], [8]. In contrast to traditional inverse control schemes, a neural network approximates an unknown nonlinear term [9]. They also have the same problems as those encountered in adaptive control methods. The third approach is to adopt the sliding mode control, which presents inherent fast response and insensitivity to plant parameter variation and/or external perturbation. In [10], a new sliding mode control law based on the measurability of all system states was presented. The law ensures global reach conditions of the sliding mode for systems subject to nonparametric time-varying uncertainties with respect to both state and input. An output feedback controller was proposed in [11] on the basis of the aforementioned approach. Sliding mode controllers essentially rely on infinite gains to achieve a good tracking performance, which is not always feasible in practice. In practice, moreover, switching will consume energy and may excite high-frequency modes. Because of these, some remedies are made [12]. The fourth approach is to adopt the nonlinear disturbance observer methods [13],[14], which are often used to tackle tracking problems for minimum-phase nonlinear systems. However, for the considered stabilizing control problem, its output is in fact all states. So, the dimension of input is unequal to that of output. Moreover, in this paper, the considered nonparametric time-varying uncertainties with respect to both state and input is neither bounded as in [13] nor twice differentiable as in [14]. Related methods can also be found in [15] and [16].

To overcome these aforementioned drawbacks, this paper proposes a stabilizing control method that involves dynamic inversion based on additive state decomposition (ASD) [17]–[19]. This paper focuses on a stabilization problem, which distinguishes it from the authors' previous work on ASD. In previous research [17] –[19], ASD has been applied to tracking problems for nonlinear systems without stabilization problems or with the stabilization problem being solved by a simple state feedback controller. The key idea of the proposed method is that, by defining a new output, nonparametric time-varying uncertainties with respect to both state and input are lumped into one disturbance by ASD at the defined output. Such a disturbance is then compensated for. The designed controller is continuous and enables state to be uniformly ultimate boundedness, and further, asymptotic stability in the presence of time-invariant uncertainties. Moreover, by choosing a special filter, the proposed controller is finally replaced by a proportional-integral (PI) controller, which is a salient feature over those methods [1]–[14]. Despite the great progress in advanced control, the PID controller remains the most popular controller. Moreover, most of devices, such as autopilots of quadrotors, offer the ports to tune the parameters of PID. It is noticed that the proposed PI controller is consistent with the controller form in [20] for a similar problem. Compared with [20], the considered plant, analysis method, and design procedure are all different, especially in the analysis method and design procedure. The bound on a key parameter, corresponding to the singular perturbation parameter in [20], is also given explicitly.

II. Problem Formulation

Consider a class of MIMO systems subject to nonparametric time-varying uncertainties with respect to both state and input as follows:

$$\dot{x}(t) = A_0 x(t) + B(h(t,u) + \sigma(t,x)), x(0) = x_0 \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the system state (taken as a measurable output), $u(t) \in \mathbb{R}^m$ is the control, $A_0 \in \mathbb{R}^{n\times n}$ is a known matrix, $B \in \mathbb{R}^{n\times m}$ is a known constant matrix, $h : [0,\infty) \times \mathbb{R}^m \to \mathbb{R}^m$ is an unknown nonlinear vector function, and $\sigma : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$ is an unknown nonlinear time-varying disturbance. For system (1), the following assumptions are made.

Assumption 1. $\operatorname{rank}\left(B, A_0 B, \ldots, A_0^{n-1} B\right) = n$.

Assumption 2. For $\forall u \in \mathbb{R}^m$ $\forall t \geq 0$, the unknown nonlinear vector function h satisfies

$$h(t,0) \equiv 0, \left\|\frac{\partial h}{\partial t}\right\| \leq l_{h_t} \|u\|$$
$$\underline{l}_{h_u} I_m \leq \frac{\partial h}{\partial u}, \left\|\frac{\partial h}{\partial u}\right\| \leq \bar{l}_{h_u}$$

where $l_{h_t}, \underline{l}_{h_u}, \bar{l}_{h_u} > 0$.

Assumption 3. The time-varying disturbance $\sigma(t,x)$ satisfies

$$\|\sigma(t,x)\| \leq k_\sigma \|x(t)\| + \delta_\sigma(t), \left\|\frac{\partial \sigma}{\partial x}\right\| \leq l_{\sigma_x}$$
$$\left\|\frac{\partial \sigma}{\partial t}\right\| \leq l_{\sigma_t} \|x(t)\| + d_\sigma(t)$$

where $\partial\sigma/\partial x \in \mathbb{R}^{m\times n}$, $k_\sigma, \delta_\sigma(t), l_{\sigma_x}, l_{\sigma_t}, d_\sigma(t) > 0$ $\forall t \geq 0$, and the ultimate bound of $\delta_\sigma(t), d_\sigma(t)$ are $b_{\delta_\sigma}, b_{d_\sigma}$, respectively.

Remark 1. By *Assumption 1*, the pair (A_0, B) is controllable. For *Assumption 2*, if $h(t,u) = (1 + 0.2\cos t)u$, then $\frac{\partial h}{\partial t} = -0.2\sin t \cdot u$ and $\frac{\partial h}{\partial u} = 1 + 0.2\cos t$. It is required that $\partial h/\partial u > \underline{l}_{h_u} I_m > 0$. Otherwise, the system (1) may be uncontrollable when $\partial h/\partial u$ is singular. Input nonlinear uncertainties inherently arise from practical actuators in realization (Example in Section IV-C) also the payload varying (Example in Section V). Similar assumptions can be also founded in [3], [9], and [10]. *Assumption 3* is also a general assumption, where the disturbance could be time-varying, bounded or state-dependent. In a word, these assumptions are made as general and close to practice as possible.

Control Objective. The control objective is to design a stabilizing controller u to drive the system state such that the state is uniformly ultimate boundedness by a small value, or $x(t) \to 0$ as $t \to \infty$ in the presence of time-invariant uncertainties. In the following, for convenience, the notation t will be dropped except when necessary for clarity.

III. ASD Dynamic Inversion Design

In this section, by ASD, the considered uncertain system is first transformed into an uncertainty-free system but subject to a lumped disturbance at the output. Then, a dynamic inversion method is applied to this transformed system. Finally, the performance of the resultant closed-loop system is analyzed.

A. Output Matrix Redefinition

By *Assumption 1*, a matrix $K \in \mathbb{R}^{n\times m}$ can be always found such that $A_0 + BK^{\mathrm{T}}$ is stable. This also implies that there exist

$0 < P, M \in \mathbb{R}^{n\times n}$ such that

$$P\left(A_0 + BK^{\mathrm{T}}\right) + \left(A_0 + BK^{\mathrm{T}}\right)^{\mathrm{T}} P = -M. \quad (2)$$

According to this, by introducing $BK^{\mathrm{T}}x - BK^{\mathrm{T}}x \equiv 0$ into (1), the system (1) is rewritten to be

$$\dot{x} = Ax + B\left(h(t,u) - K^{\mathrm{T}}x + \sigma(t,x)\right), x(0) = x_0 \quad (3)$$

where $A = A_0 + BK^{\mathrm{T}}$. Based on matrix A, a new definition of output matrix C is given in the following theorem.

Theorem 1. Under *Assumption 1*, suppose A^{T} has n negative real eigenvalues, denoted by $-\lambda_i \in \mathbb{R}$, to which n independent unit real eigenvectors correspond, denoted by $c_i \in \mathbb{R}^n$, $i = 1, \ldots, n$. If the output matrix is proposed as

$$C = \begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix} \in \mathbb{R}^{n\times m} \quad (4)$$

then

$$C^{\mathrm{T}}A = -\Lambda C^{\mathrm{T}} \quad (5)$$

where $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^{m\times m}$. Furthermore, if Λ has the form $\Lambda = \alpha I_m$, then $\det\left(C^{\mathrm{T}}B\right) \neq 0$. In particular, if $B, C \in \mathbb{R}^n$, then $C^{\mathrm{T}}B \neq 0$.

Proof. Since $c_i, i = 1, \ldots, m$ are m independent unit eigenvectors of A^{T}, it follows

$$A^{\mathrm{T}}c_i = -\lambda_i c_i, i = 1, \ldots, m$$

namely $A^{\mathrm{T}}C = -C\Lambda$. Then, $C^{\mathrm{T}}A = -\Lambda C^{\mathrm{T}}$. Next, $\det(C^{\mathrm{T}}B) \neq 0$ will be shown. Suppose, to the contrary, that $\det(C^{\mathrm{T}}B) = 0$. According to this, there exists a nonzero vector $w \in \mathbb{R}^m$ such that $w^{\mathrm{T}}C^{\mathrm{T}}B = 0$. Define $v = Cw$. Since $w \neq 0$ and C is of column full rank, it follows $v \neq 0$. With such a vector v, it further follows

$$\begin{aligned} & v^{\mathrm{T}}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \\ &= w^{\mathrm{T}}C^{\mathrm{T}}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \\ &= w^{\mathrm{T}}C^{\mathrm{T}}B\begin{bmatrix} I_m & -\alpha I_m & \cdots & (-\alpha)^{n-1} I_m \end{bmatrix} = 0. \end{aligned}$$

which implies rank $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} < n$. This contradicts *Assumption 1*. Then, $\det\left(C^{\mathrm{T}}B\right) \neq 0$. In particular, if $B, C \in \mathbb{R}^n$, then $A^{\mathrm{T}}C = -\lambda C$, i.e., $\Lambda = \lambda$ is a scale. Consequently, $C^{\mathrm{T}}B \neq 0$. □

By *Theorem 1*, a virtual output $y = C^{\mathrm{T}}x$ is defined, whose first derivative is

$$\dot{y} = -\Lambda y + C^{\mathrm{T}}B\left(h(t,u) - K^{\mathrm{T}}x + \sigma(t,x)\right) \quad (6)$$

where (5) is utilized. Then, the system (3) can be rewritten in the form [21, p. 514]

$$\begin{bmatrix} \dot{\eta} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A_\eta & B_\eta \\ 0_{m\times(n-m)} & -\Lambda \end{bmatrix}\begin{bmatrix} \eta \\ y \end{bmatrix} + \begin{bmatrix} 0_{(n-m)\times m} \\ C^{\mathrm{T}}B \end{bmatrix}\left(h(t,u) - K^{\mathrm{T}}x + \sigma(t,x)\right) \quad (7)$$

where $\eta \in \mathbb{R}^{n-m}$ is an internal part and $y \in \mathbb{R}^m$ is an external part. Since A is stable, $A_\eta \in \mathbb{R}^{(n-m)\times(n-m)}$ is stable as well, i.e., the zero dynamics are stable. Therefore, the new output matrix C makes the MIMO system from u to y be a minimum-phase system. As a result, $y(t) \to 0$ implies $x(t) \to 0$. This is why the following dynamic inversion controller design aims to make $y(t) \to 0$.

B. ASD

Consider the system (6) as the original system. The primary system is chosen as follows:

$$\dot{y}_p = -\Lambda y_p + C^{\mathrm{T}}Bu, y_p(0) = 0. \quad (8)$$

From the original system (6) and the primary system (8), we derive the following "secondary" system:

$$\begin{aligned} \dot{y} - \dot{y}_p = & -\Lambda y + C^{\mathrm{T}}B\left(h(t,u) - K^{\mathrm{T}}x + \sigma(t,x)\right) \\ & -\Lambda y_p - C^{\mathrm{T}}Bu. \end{aligned}$$

Define a new variable $y_s \in \mathbb{R}^n$ as follows:

$$y_s \triangleq y - y_p. \quad (9)$$

Then, the secondary system becomes

$$\begin{aligned} \dot{y}_s = & -\Lambda y_s + C^{\mathrm{T}}B(-u + h(t,u) - K^{\mathrm{T}}x \\ & +\sigma(t,x))y_s(0) = C^{\mathrm{T}}x_0. \end{aligned} \quad (10)$$

According to (9), it follows

$$y = y_p + y_s. \quad (11)$$

Define a transfer function

$$G(s) = (sI_m + \Lambda)^{-1} C^{\mathrm{T}}B. \quad (12)$$

Then, rearranging (8)–(11) results in

$$\begin{aligned} \dot{y}_p = & -\Lambda y_p + C^{\mathrm{T}}Bu, y_p(0) = 0 \\ y = & y_p + d_l. \end{aligned} \quad (13)$$

In this paper, $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and the inverse Laplace transform, respectively. An assumption is made on d_l.

Assumption 4. The Laplace transform of d_l exists.

Based on *Assumption 4*, the Laplace transform of d_l is

$$\begin{aligned} d_l(s) = & -G(s)u(s) + (sI_m + \Lambda)^{-1} C^{\mathrm{T}}x_0 \\ & +G(s)\mathcal{L}\left(h(t,u) - K^{\mathrm{T}}x + \sigma(t,x)\right) \end{aligned}$$

called the lumped disturbance. Furthermore, (13) is written in the form of a transfer function as

$$y(s) = G(s)u(s) + d_l(s). \quad (14)$$

The lumped disturbance d_l includes uncertainties, disturbance, and input. Fortunately, since $y_p(s) = G(s)u(s)$ and the output y are known, the lumped disturbance d_l can be obtained exactly by

$$d_l(s) = y(s) - G(s)u(s). \quad (15)$$

New Control Objective. So far, by the ASD, the control objective can be rephrased as follows: design a bounded controller

u for (14) such that the output y is uniformly ultimate boundedness by a small value, or $y(t) \to 0$ as $t \to \infty$ in the presence of time-invariant uncertainties. In (14), $G(s)$ is a known minimum-phase transfer function, while the output y and disturbance d_l are both measurable.

C. Dynamic Inversion Control

So far, by ASD, the uncertain system (1) has been transformed into an uncertainty-free system (14). For the system (14), since G is minimum-phase and known, the dynamic inversion tracking controller design is represented as follows:

$$u(s) = -G^{-1}(s)\, d_l(s). \tag{16}$$

However, the proposed controller (16) cannot be realized because the order of numerator is greater than order of denominator. So, by introducing a low-pass filter matrix $Q(s)$, the controller (16) is modified as follows:

$$u(s) = -G^{-1}(s)\, Q(s)\, d_l(s). \tag{17}$$

By employing the controller (17), the output in (14) becomes

$$y(s) = (I_m - Q(s))\, d_l(s). \tag{18}$$

Since Q is a low-pass filter matrix and the low-frequency range is often dominant in a signal, it is expected that the output will be attenuated by the transfer function $I_m - Q(s)$. A detailed analysis will be given in Section III-D.

By choosing a special filter $Q(s)$, the proposed controller (17) will be replaced by a PI controller. Substituting (15) into (17) results in

$$u(s) = -G^{-1}(s)\, Q(s)\,(y(s) - G(s)\, u(s)).$$

Then, it is further written as

$$u(s) = -G^{-1}(s)\, Q(s)\,(I_m - Q(s))^{-1} C^{\mathrm{T}} x(s). \tag{19}$$

A simple way is to choose $Q(s) = \frac{1}{\epsilon s+1} I_m$, where $\epsilon > 0$ can be considered as a singular perturbation parameter. In this case, (19) becomes

$$\begin{aligned} u(s) &= -\frac{1}{\epsilon}\left(C^{\mathrm{T}}B\right)^{-1} C^{\mathrm{T}} x(s) \\ &\quad -\frac{1}{\epsilon s}\Lambda \left(C^{\mathrm{T}}B\right)^{-1} C^{\mathrm{T}} x(s) \end{aligned} \tag{20}$$

which is a PI controller by taking $C^{\mathrm{T}}x$ as the input. If $\det\left(C^{\mathrm{T}}B\right) \neq 0$, then the aforementioned controller is realizable.

Remark 2. By the proposed output matrix redefinition, the considered MIMO uncertain system is transformed into a minimum-phase MIMO system with the relative degree 1. By ASD, it is further transformed into a first-order transfer function subject to a lumped disturbance. Owing to the simple transfer function, the controller design for (14) is straightforward by the idea of dynamic inversion.

Remark 3. The output matrix can also be defined as $\bar{C} = PB$, where P satisfies (2). It can lead to a positive real transfer function $\bar{G}(s) = \bar{C}^{\mathrm{T}}(sI_n - A)^{-1} B$, which is minimum-phase with the relative degree 1. While, the proposed $G(s)$in (12) is minimum phase with the relative degree 1. Moreover, its element is a first-order system. This makes the resulting controller have a simple PI form if $Q(s) = \frac{1}{\epsilon s+1} I_m$ is adopted. Surely, the filter matrix $Q(s)$ can also be chosen as another first-order one or a higher order one. However, the simple PI form cannot be obtained.

D. Stability Analysis

Since the lumped disturbance d_l involves the input u, the resultant closed-loop system may be unstable. Next, some conditions are given to guarantee that the control input u is bounded. Substituting d_l into (17) results in

$$\begin{aligned} u(s) &= G^{-1}(s)\, Q(s)\, G(s)\left[\mathcal{L}\left(-h(t,u) + K^{\mathrm{T}}x - \sigma(t,x)\right)\right. \\ &\quad \left.+u(s)\right] - G^{-1}(s)\, Q(s)\,(sI_m + \Lambda)^{-1} C^{\mathrm{T}} x_0. \end{aligned} \tag{21}$$

Let the filter $Q(s) = \frac{1}{\epsilon s+1} I_m$. Then, (21) is further written in the time domain as

$$\epsilon \dot{u} = -h(t,u) + K^{\mathrm{T}}x - \sigma(t,x) + \xi(t) \tag{22}$$

where $\xi(t) = \mathcal{L}^{-1}(-(C^{\mathrm{T}}B)^{-1}C^{\mathrm{T}}x_0)$ will vanish. The following theorem will give an explicit bound on ϵ, below which the stability of closed-loop dynamics forming by (3) and (22) can be guaranteed.

Theorem 2. Suppose 1) *Assumptions 1–4* hold, 2) the controller is designed as (17) with $Q(s) = \frac{1}{\epsilon s+1} I_m$, 3) ϵ satisfies

$$0 < \epsilon < \frac{\underline{l}_{h_u}}{\gamma_1 + \frac{2}{\gamma_0}\gamma_2^2} \tag{23}$$

where

$$\begin{aligned} \gamma_0 &= \lambda_{\min}(M) \\ \gamma_1 &= 2\left(\|K\| + l_{\sigma_x}\right)\|B\| + 2\frac{l_{h_t}}{\underline{l}_{h_u}} \end{aligned} \tag{24}$$

$$\begin{aligned} \gamma_2 &= \|P\|\,\|B\| + \|A\|\left(\|K\| + l_{\sigma_x}\right) \\ &\quad + \frac{l_{h_t}}{\underline{l}_{h_u}}\left(\|K\| + k_\sigma\right) + l_{\sigma_t} \end{aligned} \tag{25}$$

and matrices P, M satisfy (2). Then, for any initial condition x_0, the state of system (1) satisfies[1]

$$\|x(t)\| \to \mathcal{B}\left(\sqrt{\frac{\epsilon}{\lambda_{\min}(P)\,\eta(\epsilon)\,\underline{l}_{h_u}}}\left(\frac{l_{h_t}}{\underline{l}_{h_u}} b_{\delta_\sigma} + b_{d_\sigma}\right)\right)$$

where $\eta(\epsilon) = \min(\frac{\gamma_0}{2\lambda_{\max}(P)}, \frac{1}{\epsilon}\underline{l}_{h_u} - \gamma_1 - \frac{2}{\gamma_0}\left(\gamma_2 + l_{\sigma_t}\right)^2)$. Furthermore, if $l_{h_t}\delta_\sigma(t) \to 0$ and $d_\sigma(t) \to 0$, then, for any initial condition x_0, the state $\|x(t)\| \to 0$ as $t \to \infty$.

Proof. See *Appendix A*.

Remark 4. The bound on the parameter ϵ, namely (23), is given explicitly. The condition $l_{h_t} = 0$ implies that the function h is time invariant. Also, $\delta_\sigma(t) \to 0$ and $d_\sigma(t) \to 0$ implies that $\|\sigma(t,x)\| \to 0$ and $\left\|\frac{\partial \sigma}{\partial t}\right\| \to 0$ as $x \to 0$. If (1) is time invariant, then $l_{h_t}\delta_\sigma(t) \equiv 0$ and $d_\sigma(t) \equiv 0$. ■

[1] $\mathcal{B}(\delta) \triangleq \{\xi \in \mathbb{R}\,|\,\|\xi\| \leq \delta\}$, $\delta > 0$. The notation $z(t) \to \mathcal{B}(\delta)$ means $\min_{y\in\mathcal{B}(\delta)} |z(t) - y| \to 0$ as $t \to \infty$.

Procedure	
Step 1	*Choose* n *negative real eigenvalues* $-\lambda_i < 0$, $i = 1, \cdots, n$. *By pole placement, design* K *such that* the eigenvalues of $A = A_0 + BK^{\mathrm{T}}$ *matches* $-\lambda_i$, $i = 1, \ldots, n$.
Step 2	*Define* $C = [\, c_1 \quad \cdots \quad c_m \,] \in \mathbb{R}^{n \times m}$, *where* $c_i \in \mathbb{R}^n$ *are independent unit eigenvectors corresponding to eigenvalues* $-\lambda_i$ *of* A^{T}, $i = 1, \ldots, m$ *,respectively.* If $\det(C^{\mathrm{T}}B) = 0$, then goto *Step 1* *to choose* $-\lambda_i = \alpha < 0$, $i = 1, \ldots, n$.
Step 3	*Design (20) with* $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$.
Step 4	*Choose an appropriate* $\epsilon > 0$.

Remark 5. The condition (23) implies the existence of ϵ. In practice, only the parameter ϵ needs to be tuned, so the parameters $l_{h_t}, \underline{l}_{h_u}, \overline{l}_{h_u}, k_\sigma, \delta_\sigma, l_{\sigma_x}, l_{\sigma_t}, d_\sigma$ need not be known.

From the aforementioned analysis, the design procedure is summarized as follows. It should be noted that $\det(C^{\mathrm{T}}B) \neq 0$ if $-\lambda_i = \alpha$, $i = 1, \ldots, n$. This is guaranteed by *Theorem 1*. Therefore, *Step 2* will only fail once at most.

IV. Numerical Simulations

To demonstrate its effectiveness, the proposed control method is applied to a simple example, two existing problems in [1] and [10] for comparison by numerical simulations.

A. Simple Example

Consider a simple uncertain system

$$\dot{x}(t) = x(t) + (au(t) + bx(t) + c), x(0) = 0 \tag{26}$$

which can be expressed as (1) with $A_0 = B = 1$, $h(t,u) = au(t)$, and $\sigma(t,x) = bx(t) + c$. Here, $a, b, c \in \mathbb{R}$ are unknown constants, but the sign of a is known. Without loss of generality, it is assumed $a > 0$. In this example, it is assumed that $a = b = c = 1$. The control objective is to design a stabilizing controller to drive the system state $x(t) \to 0$ as $t \to \infty$. In the following, two control methods are presented here. The first one is a classical adaptive control, the other is the proposed PI control method. The adaptive controller is designed as

$$\begin{aligned} u_{\text{adp}} &= \hat{a}_{\text{inv}} v \\ v &= -2x - \hat{b}x - \hat{c} \end{aligned} \tag{27}$$

with adaptive laws

$$\dot{\hat{a}}_{\text{inv}} = -xv, \dot{\hat{b}} = x^2, \dot{\hat{c}} = x. \tag{28}$$

The adaptive controller can drive the system state $x(t) \to 0$ as $t \to \infty$ (See *Appendix B*). For the dynamics (26), according to the design procedure at the end of *Section III-D*, the following design is given. For the scale system, the design becomes very simple.

1) *Step 1*: According to the procedure, design $K = -2$ resulting in $A = -1$.

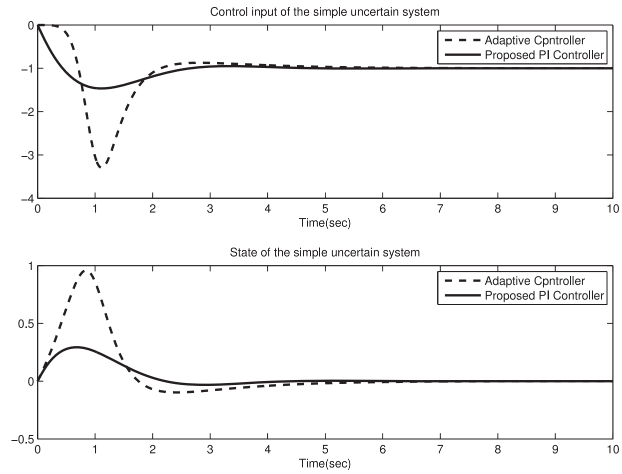

Fig. 1. Stabilization performance of an adaptive controller and the proposed PI controller.

2) *Step 2*: Then, selecting eigenvectors of A^{T} corresponding to its eigenvalue -1 results in $C = 1$.

3) *Step 3*: According to (20), design the PI controller

$$u_{\text{PI}} = -\frac{1}{\epsilon}x - \frac{1}{\epsilon s}x.$$

4) *Step 4*. For comparison[2], choose $\epsilon = 1/3$.

The performance of the two controller is shown in Fig. 1. From it, both controllers u_{adp} and u_{PI} drive the system state $x(t) \to 0$ as $t \to \infty$, but u_{PI} outperforms u_{adp} because of only one integrator used in u_{PI}. From the controller structure, it is easy to see that the proposed PI controller is also simple.

B. Uncertain SISO System

As in [10], the following uncertain dynamics are considered:

$$\dot{x}(t) = A_0 x(t) + B(\Phi(u(t)) + e(x)) \tag{29}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \Phi(u(t)) &= \left(0.5 + 0.3 \sin u(t) + e^{0.2|\cos u(t)|}\right) u(t) \\ e(x) &= (0.3 + 0.2 \cos x_1) \sqrt{x_1^2 + x_2^2 + x_3^2} - 0.5 \sin x_2. \end{aligned}$$

The objective is to drive $x(t) \to 0$ as $t \to \infty$. For the dynamics (29), according to the design procedure at the end of Section III-D, the following design is given.

[2]For adaptive control, if the estimates are equal to the unknown parameters, respectively, then the closed-loop dynamics becomes $\dot{x}(t) = -x(t)$. For the proposed PI controller, the term $-\frac{1}{\epsilon}x$ is used to stabilize, while the left term $-\frac{1}{\epsilon s}x$ is used to compensate for the constant disturbance. Therefore, for comparison, choose $\epsilon = 1/3$ so that the closed-loop dynamics by the proposed PI controller becomes $\dot{x}(t) = -x(t)$ as well.

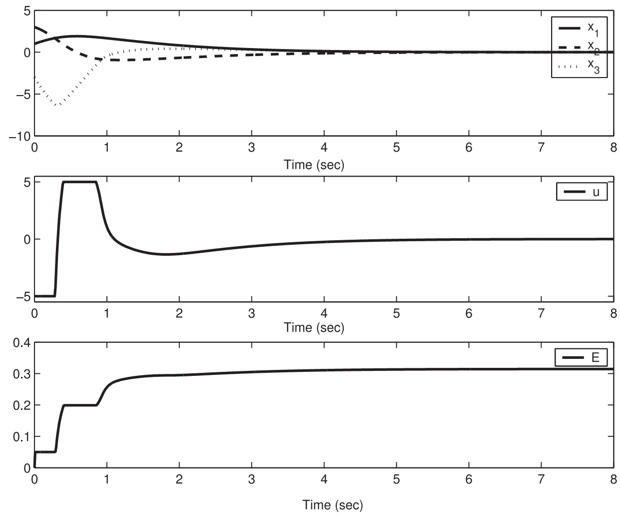

Fig. 2. Stabilization performance of the SISO dynamics driven by the ASD dynamic inversion stabilizing controller.

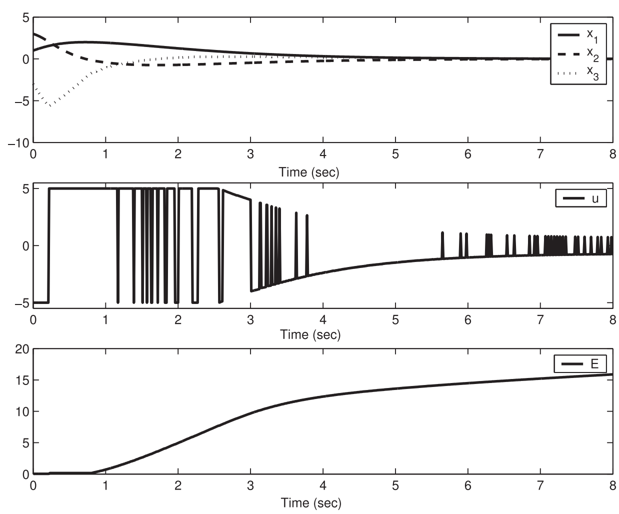

Fig. 3. Stabilization performance of the SISO dynamics driven by the sliding mode controller proposed in [10].

1) *Step 1*: According to the procedure, design $K = [-5 \quad -8 \quad -5]^{\mathrm{T}}$ resulting in $A = A_0 + BK^{\mathrm{T}}$ with three different negative real eigenvalues $-1, -2, -3$.
2) *Step 2*: Selecting eigenvectors of A^{T} corresponding to its eigenvalue -1 results in $C = [6 \quad 5 \quad 1]^{\mathrm{T}}$. Obviously, $C^{\mathrm{T}}B \neq 0$.
3) *Step 3*: According to (20), design the PI controller $u = -\frac{1}{\epsilon}C^{\mathrm{T}}x - \frac{1}{\epsilon s}C^{\mathrm{T}}x$.
4) *Step 4*: *Choose* $\epsilon = 0.1$.

The range of the control input u is chosen as $[-5, 5]$ in practice. Driven by the designed controller, the control performance is shown in Fig. 2. As shown, all states converge to zero. Moreover, the control input is continuous and bounded. The control performance by the sliding mode controller proposed in [10] is shown in Fig. 3. The index $E(t) = \int_0^t |\dot{u}(s)| ds$ is introduced to represent the energy cost. It is easy to observe that our proposed controller saves more energy compared with the sliding mode controller proposed in [10].

C. Uncertain MIMO System

As in [1], the lateral/directional baseline model of an F-16 from [22, pp. 584–592] flying at sea level with an airspeed of 502 ft/s and angle of attack of 2.11° is used. Denote the angle of sideslip, the roll angle, the stability axis roll and yaw rates, aileron and rudder control by $\beta, \phi, p_s, r_s, \delta_a$, and δ_r, respectively. The full roll/yaw dynamics in state-space form are

$$\underbrace{\begin{bmatrix} \dot{\beta}(t) \\ \dot{\phi}(t) \\ \dot{p}_s(t) \\ \dot{r}_s(t) \end{bmatrix}}_{\dot{x}(t)} = A_0 \underbrace{\begin{bmatrix} \beta(t) \\ \phi(t) \\ p_s(t) \\ r_s(t) \end{bmatrix}}_{x(t)} + B \begin{bmatrix} \delta_a(t) + f_1(\beta(t), p_s(t), r_s(t), \delta_a(t)) \\ \delta_r(t) + f_2(\beta(t), p_s(t), r_s(t), \delta_r(t)) \end{bmatrix}. \tag{30}$$

Here

$$A_0 = \begin{bmatrix} -0.3220 & 0.064 & 0.0364 & -0.9917 \\ 0 & 0 & 1 & 0.0393 \\ -30.6490 & 0 & -3.6784 & 0.6646 \\ 8.5395 & 0 & -0.0254 & -0.4764 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.7331 & 0.1315 \\ -0.0319 & -0.0620 \end{bmatrix}$$

$$\begin{aligned} f_1 =& \left((1 - C_1)e^{-\frac{(\beta-\beta_0)^2}{2\sigma_1^2}} + C_1\right)(\tanh(\delta_a + h_1) \\ &+ \tanh(\delta_a - h_1) + 0.001\delta_a) \\ &+ D_1\cos(A_1 p_s - \omega_1)\sin(A_2 r_s - \omega_2) + D_2 \\ f_2 =& \left((1 - C_2)e^{-\frac{(\beta-\beta_0)^2}{2\sigma_2^2}} + C_2\right)(\tanh(\delta_r + h_2) \\ &+ \tanh(\delta_r - h_2) + 0.001\delta_r) \\ &+ D_3\cos(A_3 p_s - \omega_3)\sin(A_4 r_s - \omega_4) + D_4 \end{aligned}$$

where $A_1 = 0.33$, $A_2 = 0.195$, $A_3 = 0.45$, $A_4 = 1.85$, $D_1 = 0.295$, $D_2 = -0.0865$, $D_3 = 0.055$, $D_4 = -0.007$, $w_1 = 1.6$, $w_2 = 0$, $w_3 = -1.9$, $w_4 = 0$, $C_1 = 0.3$, $C_2 = 0.3$, $h_1 = 7$, $h_2 = 2.7$, $\delta_1 = 0.25$, $\delta_2 = 0.25$, and $\beta_0 = 0$. Readers are suggested to refer to [1] for details. The dynamics (30) can be formulated as (1). The objective is to drive $x(t) \to 0$ as $t \to \infty$.

For the dynamics (30), according to the design procedure at the end of Section III-D, the following design is given.

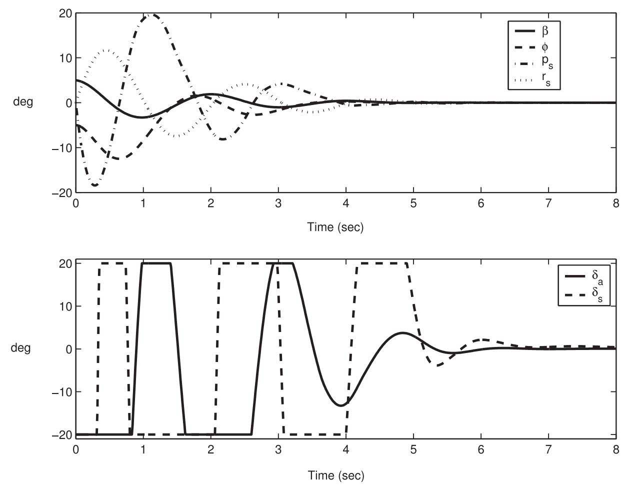

Fig. 4. Stabilization performance of F-16 Roll/Yaw dynamics driven by the ASD dynamic inversion stabilizing controller.

1) *Step 1*: According to the procedure, design

$$K = \begin{bmatrix} -27.5037 & 93.4020 \\ 14.2953 & 35.0244 \\ 4.5010 & 13.9005 \\ 12.7039 & 58.8096 \end{bmatrix}$$

resulting in $A = A_0 + BK^{\mathrm{T}}$ with the four negative real eigenvalues $-1, -2, -3, -4$.

2) *Step 2*: Selecting eigenvectors of A^{T} corresponding to its eigenvalues $-1, -2$ results in

$$C = \begin{bmatrix} 0.6537 & -0.5473 \\ 0.6819 & 0.7985 \\ 0.2285 & 0.1961 \\ -0.2354 & 0.1561 \end{bmatrix}.$$

Obviously, $\det\left(C^{\mathrm{T}}B\right) \neq 0$.

3) *Step 3*: Design the PI controller

$$u = \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} = -\left(C^{\mathrm{T}}B\right)^{-1} \begin{bmatrix} \frac{s+1}{\epsilon s} & 0 \\ 0 & \frac{s+2}{\epsilon s} \end{bmatrix} C^{\mathrm{T}}x. \quad (31)$$

4) *Step 4*: Choose the appropriate $\epsilon = 0.2$.

The range of the control input δ_a, δ_r is chosen as $[-20°, 20°]$ in practice. Driven by (31), the control performance is shown in Fig. 4. As shown, all states converge to zero. Moreover, the control input is continuous and bounded. Here, the information of nonlinear terms f_1 and f_2 are not required, let alone learn the parameters. So, compared with only the stabilizing control term[3] in [1], the proposed controller design and controller structure are both simpler.

V. Application: Attitude Control of A Quadrotor

Quadrotor control has been an active area [23] –[25]. In this section, in order to show its practicability, the proposed ASD dynamic inversion design is applied to attitude control of a real quadrotor when its inertia moment matrix is subject to a large uncertainty.

A. Problem Formulation

By taking actuator dynamics into account, the linear roll model of the quadrotor around hover conditions is

$$\underbrace{\begin{bmatrix} \dot{\phi}(t) \\ \dot{p}(t) \\ \dot{L}(t) \end{bmatrix}}_{\dot{x}_\phi} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\omega \end{bmatrix}}_{A_0} \underbrace{\begin{bmatrix} \phi(t) \\ p(t) \\ L(t) \end{bmatrix}}_{x_\phi} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}}_{B_0} J_\phi^{-1}\tau_\phi \quad (32)$$

where ω is the actuator bandwidth, τ_ϕ is the command torque to be selected, and $x_\phi = [\phi \quad p \quad L]^{\mathrm{T}} \in \mathbb{R}^3$ with $\phi, p, L \in \mathbb{R}$ being the angle, angular velocity, and torque of the roll channel in the body-fixed frame, respectively. In practice, ϕ and p are measurable but L cannot. According to this, L is estimated by $\dot{\hat{L}} = -\omega\hat{L} + \omega J_\phi^{-1}\tau_\phi, \hat{L}(0) = 0$, taken as the true measurement for simplicity. In this sense, x_ϕ is known. Let $J \in \mathbb{R}^{3\times 3}$ be the inertia moment matrix of the quadrotor, $x_\theta = [\theta \quad q \quad M]^{\mathrm{T}} \in \mathbb{R}^3$ and $x_\psi = [\psi \quad r \quad N]^{\mathrm{T}} \in \mathbb{R}^3$. Here θ and $\psi \in \mathbb{R}$ are pitch and yaw angle, respectively, while q, r are, respectively, their angular velocity in the body-fixed frame. The torques $M, N \in \mathbb{R}$ represent the airframe pitch and yaw torque, respectively. Similar to (32), the linear attitude model of the quadrotor is expressed as

$$\dot{x} = \bar{A}_0 x + BJ^{-1}\tau, x(0) = x_0. \quad (33)$$

Here, $x = [x_\phi \quad x_\theta \quad x_\psi]^{\mathrm{T}} \in \mathbb{R}^9$ is the state and $\tau = [\tau_\phi \quad \tau_\theta \quad \tau_\psi]^{\mathrm{T}} \in \mathbb{R}^3$. The system matrix and input matrix in (33) are $\bar{A}_0 = \mathrm{diag}(A_0, A_0, A_0) \in \mathbb{R}^{9\times 9}$ and $B = \mathrm{diag}(B_0, B_0, B_0) \in \mathbb{R}^{9\times 3}$. In practice, the inertia moment matrix $J \in \mathbb{R}^{3\times 3}$, related to the position and weight of payload such as instruments and batteries, is often difficult to determine. Moreover, the payload of the quadrotor is often time-varying owing to fuel consumption or pesticide spraying. Therefore, compared with the true inertia moment matrix, the real inertia moment matrix J may have a large uncertainty. Assume the nominal inertia moment matrix to be $J_0 \in \mathbb{R}^{3\times 3}$. By employing it, a controller is designed to stabilize the attitude (33) as

$$\tau = J_0\left(\bar{K}^{\mathrm{T}}x + u\right) \quad (34)$$

where $\bar{K} \in \mathbb{R}^{9\times 3}$ and u will be specified later. Then, (33) can be cast in the form of (1), where $A_0 = \bar{A}_0 + B\bar{K}^{\mathrm{T}}$, $h(t, u) = J^{-1}J_0 u$, and $\sigma(t, x) = \left(J^{-1}J_0 - I_3\right)\bar{K}^{\mathrm{T}}x$. To accord with the *Step 1* of the proposed control procedure, choose

$$\bar{K} = \mathrm{diag}\left(K_0, K_0, K_0\right), K_0 = \left[-3.0 \quad -4.2 \quad -0.27\right]^{\mathrm{T}}.$$

The resultant $A_0 = \bar{A}_0 + B\bar{K}^{\mathrm{T}}$ is stable with $\det\left(sI_9 - A_0\right) = (s+15)^3 (s+3)^3 (s+1)^3$.

B. ASD Dynamic Inversion Control

For system (33), according to the design procedure at the end of Section III-D, the following design steps are given.

[3]A tracking controller is designed in [1], which includes a stabilizing control term for error dynamics.

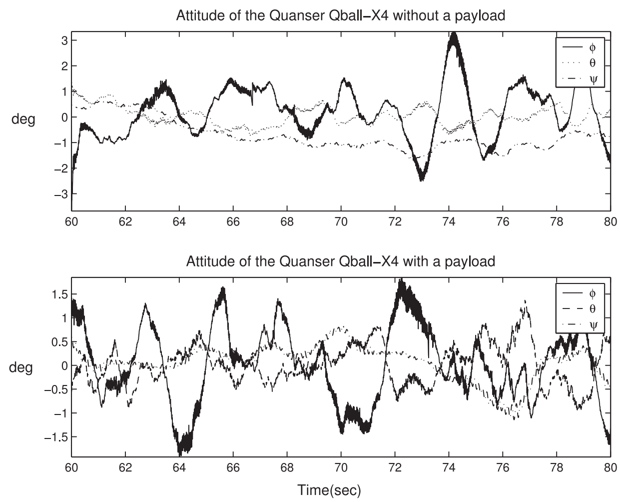

Fig. 5. Attitude stabilization performance of the ASD dynamic inversion stabilizing controller.

1) *Step 1:* Thanks to the aforementioned designed A_0, *Step 1* is skipped. As a result, the matrix $A = A_0$ is stable and $\det(sI_9 - A) = (s+15)^3 (s+3)^3 (s+1)^3$.
2) *Step 2*: The output matrix C is redefined by

$$\begin{aligned} C &= \text{diag}(C_0, C_0, C_0), \\ C_0 &= \begin{bmatrix} 0.9283 & 0.3713 & 0.0206 \end{bmatrix}^{\mathrm{T}} \end{aligned} \tag{35}$$

with $C^{\mathrm{T}}A = -C^{\mathrm{T}}$. Obviously, $C^{\mathrm{T}}B \neq 0$.
3) *Step 3:* Design the PI controller

$$u = -\frac{1}{\epsilon}\left(C^{\mathrm{T}}B\right)^{-1} C^{\mathrm{T}}x - \frac{1}{\epsilon s}\left(C^{\mathrm{T}}B\right)^{-1} C^{\mathrm{T}}x. \tag{36}$$

4) *Step 4:* Choose the appropriate $\epsilon = 0.2$.

C. Experiment

The experiment is performed on a Quanser Qball-X4, a quadrotor developed by the Quanser company. Its nominal inertia matrix is J_0 =diag$(0.03, 0.03, 0.04)$ kg·m^2 and the actuator bandwidth is $\omega = 15$ rad/s. The proposed controller (34) is used only for attitude control, while an existing position controller offered by the Quanser company is retained. By using them, a hover control is expected to perform for the Quanser Qball-X4. The experimental results are shown in Fig. 5. Then, in order to demonstrate the effectiveness of the proposed method, a 0.145-kg payload is attached to the 1.4-kg Quanser Qball-X4 to change its inertia moment matrix, shown in Fig. 6. With the same controller, the experimental results are shown in Fig. 5. As shown, the proposed controller is robust against the uncertainty in the inertia moment matrix. The video is available at https://www.youtube.com/watch?v=XE4plSkHYxc and http://rfly.buaa.edu.cn/. By using this control scheme, varying payload can also be handled. Another application to stabilizing control of a hexacopter subject to unknown propeller damages can be found in [27].

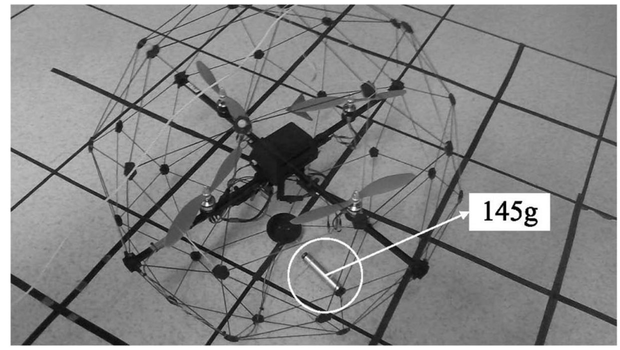

Fig. 6. Quanser Qball-X4 is attached a 0.145-kg payload.

VI. Conclusion

Stabilizing control for a class of MIMO systems is considered subject to nonparametric time-varying uncertainties with respect to both state and input. This study has three contributions: 1) an ASD dynamic inversion stabilizing controller design, which can solve the stabilization problem for a class of uncertain MIMO systems, 2) a PI controller is designed with fewer tuning parameters, 3) the definition of a new output matrix, which transforms uncertain MIMO systems into a first-order MIMO system. The proposed controller design is applied to two existing problems by numerical simulations and the attitude control problem for a quadrotor by a real experiment. From the two simulations and the experiment, it is observed that: 1) the proposed controller is robust against uncertainties; 2) the design and controller structure are both simpler; 3) proposed controller is continuous, so it can save energy compared with the sliding mode controller.

Appendix

A. Proof of Theorem 2

The following preliminary result is needed.

Lemma 1 [26, p.74.]. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable in an open convex set $D \subset \mathbb{R}^n$. For any $x, x+p \in D$, $F(x+p) - F(x) = \int_0^1 \frac{\partial F}{\partial z}|_{z=x+sp} \mathrm{d}s \cdot p$.

Denote $v = h(t,u) - K^{\mathrm{T}}x + \sigma(t,x)$. Then, the system (3) becomes

$$\dot{x} = Ax + Bv$$

and the derivative of $\epsilon\dot{v}$ is calculated to be

$$\begin{aligned} \epsilon\dot{v} &= \frac{\partial h}{\partial u}\epsilon\dot{u} + \epsilon\frac{\partial h}{\partial t} - \epsilon K^{\mathrm{T}}\dot{x} + \frac{\partial \sigma}{\partial x}\epsilon\dot{x} + \epsilon\frac{\partial \sigma}{\partial t} \\ &= \frac{\partial h}{\partial u}(-v+\xi) + \epsilon\left(-K^{\mathrm{T}} + \frac{\partial \sigma}{\partial x}\right)(Ax+Bv) \\ &\quad + \epsilon\frac{\partial h}{\partial t} + \epsilon\frac{\partial \sigma}{\partial t} \;[(22) \text{ is used}] \\ &= \left[-\frac{\partial h}{\partial u} + \epsilon\left(-K^{\mathrm{T}} + \frac{\partial \sigma}{\partial x}\right)B\right]v + \epsilon\frac{\partial h}{\partial t} \\ &\quad + \epsilon\frac{\partial \sigma}{\partial t} + \frac{\partial h}{\partial u}\xi + \epsilon\left(-K^{\mathrm{T}} + \frac{\partial \sigma}{\partial x}\right)Ax. \end{aligned}$$

Consequently, the closed-loop dynamics (22) and (3) are

$$\dot{x} = Ax + Bv$$
$$\dot{v} = \left[-\frac{1}{\epsilon}\frac{\partial h}{\partial u} + \left(-K^{\mathrm{T}} + \frac{\partial \sigma}{\partial x}\right)B\right]v + \frac{\partial h}{\partial t} + \frac{\partial \sigma}{\partial t} + \frac{1}{\epsilon}\frac{\partial h}{\partial u}\xi + \left(-K^{\mathrm{T}} + \frac{\partial \sigma}{\partial x}\right)Ax. \quad (37)$$

Choose a candidate Lyapunov function as follows:

$$V = x^{\mathrm{T}}Px + v^{\mathrm{T}}v$$

where $0 < P \in \mathbb{R}^{n\times n}$ satisfies (2). Taking the derivative of V along the solution of (37) yields

$$\dot{V} = x^{\mathrm{T}}\left(PA + A^{\mathrm{T}}P\right)x + 2v^{\mathrm{T}}\left(\frac{\partial h}{\partial t} + \frac{\partial \sigma}{\partial t} + \frac{1}{\epsilon}\frac{\partial h}{\partial u}\xi\right) - 2v^{\mathrm{T}}\left[\frac{1}{\epsilon}\frac{\partial h}{\partial u} - \left(-K^{\mathrm{T}} + \frac{\partial \sigma}{\partial x}\right)B\right]v + 2x^{\mathrm{T}}\left[PB - A^{\mathrm{T}}\left(-K^{\mathrm{T}} + \frac{\partial \sigma}{\partial x}\right)^{\mathrm{T}}\right]v.$$

By (2), it follows $x^{\mathrm{T}}\left(PA + A^{\mathrm{T}}P\right)x \leq -\gamma_0 \|x\|^2$, where $\gamma_0 = \lambda_{\min}(M)$. Then, by *Assumptions 2* and *3*,

$$\dot{V} \leq 2\|v\|\left(l_{h_t}\|u\| + l_{\sigma_t}\|x\| + d_\sigma(t) + \frac{\bar{l}_{h_u}}{\epsilon}\|\xi(t)\|\right) - \gamma_0\|x\|^2 - 2\left(\frac{1}{\epsilon}\underline{l}_{h_u} - \frac{1}{2}\gamma_1'\right)\|v\|^2 + 2\gamma_2'\|v\|\|x\| \quad (38)$$

where $\gamma_1' = 2(\|K\| + l_{\sigma_x})\|B\|$ and $\gamma_2' = \|P\|\|B\| + \|A\|(\|K\| + l_{\sigma_x})$. Next, the relationship between $\|u\|$ and $\|v\|$ needs to be derived to eliminate $\|u\|$ in (38). By *Lemma 1*, it follows

$$h(t,u) = h(t,0) + \left(\int_0^1 \left.\frac{\partial h}{\partial z}\right|_{z=su} \mathrm{d}s\right)u.$$

Furthermore, since $v = h(t,u) - K^{\mathrm{T}}x + \sigma(t,x)$, it follows

$$u^{\mathrm{T}}\left(\int_0^1 \left.\frac{\partial h}{\partial z}\right|_{z=su}\mathrm{d}s\right)u = u^{\mathrm{T}}\left(v + K^{\mathrm{T}}x - \sigma(t,x) - h(t,0)\right).$$

Further by *Assumptions 2* and *3*, the aforementioned equation becomes

$$\underline{l}_{h_u}\|u\|^2 \leq \left\|u^{\mathrm{T}}\left(v + K^{\mathrm{T}}x - \sigma(t,x) - h(t,0)\right)\right\| \leq \|u\|\left(\|v\| + \|K\|\|x\| + \|\sigma(t,x)\|\right).$$

This implies

$$\|u\| \leq \frac{1}{\underline{l}_{h_u}}\left(\|v\| + \|K\|\|x\| + \|\sigma(t,x)\|\right).$$

With the aforementioned inequality and the following inequality:

$$2\|v\|\left(\frac{l_{h_t}}{\underline{l}_{h_u}}\delta_\sigma(t) + d_\sigma(t) + \frac{\bar{l}_{h_u}}{\epsilon}\|\xi\|\right) \leq \frac{1}{\epsilon}\underline{l}_{h_u}\|v\|^2 + \frac{\epsilon}{\underline{l}_{h_u}}\left(\frac{l_{h_t}}{\underline{l}_{h_u}}\delta_\sigma(t) + d_\sigma(t) + \frac{\bar{l}_{h_u}}{\epsilon}\|\xi\|\right)^2$$

the inequality (38) becomes

$$\dot{V} \leq -\gamma_0\|x\|^2 - \left(\frac{1}{\epsilon}\underline{l}_{h_u} - \gamma_1\right)\|v\|^2 + \frac{\epsilon}{\underline{l}_{h_u}}\left(\frac{l_{h_t}}{\underline{l}_{h_u}}\delta_\sigma(t) + d_\sigma(t) + \frac{\bar{l}_{h_u}}{\epsilon}\|\xi(t)\|\right)^2 + 2\gamma_2\|v\|\|x\|$$

where γ_1 and γ_2 are defined in (25). Since the inequality $2ab \leq a^2 + b^2$ always holds, then

$$2\gamma_2\|v\|\|x\| \leq \frac{\gamma_0}{2}\|x\|^2 + \frac{2}{\gamma_0}\gamma_2^2\|v\|^2.$$

Consequently

$$\dot{V} \leq -\frac{\gamma_0}{2}\|x\|^2 + \frac{\epsilon}{\underline{l}_{h_u}}\left(\frac{l_{h_t}}{\underline{l}_{h_u}}\delta_\sigma(t) + d_\sigma(t) + \frac{\bar{l}_{h_u}}{\epsilon}\|\xi(t)\|\right)^2 - \left(\frac{1}{\epsilon}\underline{l}_{h_u} - \gamma_1 - \frac{2}{\gamma_0}\gamma_2^2\right)\|v\|^2.$$

Furthermore

$$\dot{V} \leq -\eta(\epsilon)V + \frac{\epsilon}{\underline{l}_{h_u}}\left(\frac{l_{h_t}}{\underline{l}_{h_u}}\delta_\sigma(t) + d_\sigma(t) + \frac{\bar{l}_{h_u}}{\epsilon}\|\xi(t)\|\right)^2$$

where $\eta(\epsilon) = \min(\frac{\gamma_0}{2\lambda_{\max}(P)}, \frac{1}{\epsilon}\underline{l}_{h_u} - \gamma_1 - \frac{2}{\gamma_0}\gamma_2^2)$. If (23) is satisfied, then $\eta(\epsilon) > 0$. Since $\xi(t) \to 0$ as $t \to \infty$ for any initial condition x_0 and the ultimate bound of $\delta_\sigma(t), d_\sigma(t)$ are $b_{\delta_\sigma}, b_{d_\sigma}$, respectively, $\|x(t)\| \to \mathcal{B}(\sqrt{\frac{\epsilon}{\lambda_{\min}(P)\eta(\epsilon)\underline{l}_{h_u}}}(\frac{l_{h_t}}{\underline{l}_{h_u}}b_{\delta_\sigma} + b_{d_\sigma}))$. Furthermore, if $l_{h_t}\delta_\sigma(t) \to 0$ and $d_\sigma(t) \to 0$, then the state $\|x(t)\| \to 0$ as $t \to \infty$.

B. Stability Proof for Adaptive Control

Substituting (27) into (26) yields

$$\dot{x} = -x + (a\hat{a}_{\text{inv}} - 1)v + \left(b - \hat{b}\right)x + (c - \hat{c}). \quad (39)$$

Design a Lyapunov function as

$$V = \frac{1}{2}x^2 + \frac{1}{2a}(a\hat{a}_{\text{inv}} - 1)^2 + \frac{1}{2}\left(b - \hat{b}\right)^2 + \frac{1}{2}(c - \hat{c})^2.$$

Then, taking the derivative of V along the solution of (39) results in

$$\dot{V} = -x^2 + (a\hat{a}_{\text{inv}} - 1)xv + \left(b - \hat{b}\right)x^2 + (c - \hat{c})x + (a\hat{a}_{\text{inv}} - 1)\dot{\hat{a}}_{\text{inv}} - \left(b - \hat{b}\right)\dot{\hat{b}} - (c - \hat{c})\dot{\hat{c}}. \quad (40)$$

Substituting adaptive laws (28) into (40) results in $\dot{V} = -x^2$. According to Invariance-like Theorems [21, pp. 322–329], $x(t) \to 0$ as $t \to \infty$.

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