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A new generator of causal ideal internal dynamics for a class of unstable linear differential equations

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SUMMARY

The generator design for causal ideal internal dynamics (IID), namely, solving IID, is a fundamental problem in a nonminimum-phase output tracking process. In this paper, for a class of unstable matrix differential equations, a new causal IID generator is proposed, whose parameters are partly chosen via $\mathcal{H}_2/\mathcal{H}_\infty$ optimization. Compared with existing similar design schemes, it is applicable to matrix differential equations with singular system matrices. Also, it requires less computation, avoids taking higher order derivatives, and can be easily extended to treat slowly time-varying matrix differential equations without the need for extra computation. Copyright © 2016 John Wiley & Sons, Ltd.

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KEY WORDS: nonminimum-phase systems; ideal internal dynamics; causal case; tracking

1. INTRODUCTION

A system is nonminimum phase if its internal dynamics (ID) are unstable [1]. Nonminimum-phase output tracking is a challenging, real-life control problem that has been extensively studied [2–6]. An important solution to this problem is to identify the state references such that the output tracking problem can be converted into a stabilization problem, which can be solved by using conventional control methods, such as sliding mode control [7–9]. State references are composed of output references and internal state references. The former are often given, whereas the latter is difficult to obtain for an unstable ID, namely, for a nonminimum-phase system. A bounded solution to the unstable ID is called the ideal internal dynamics (IID) [7]. A basic *IID problem* can be stated as follows:

IID problem: Given $\xi \in \mathcal{L}_\infty([0,\infty),\mathbb{R})$‡, $A \in \mathbb{R}^{n\times n}$, *and* $N \in \mathbb{R}^n$, *find an initial condition* η_0 *such that the solution* $\eta(t)$ *to the following differential equation*

$$\dot{\eta}(t) = A\eta(t) + N\xi(t), \eta(0) = \eta_0, t \geq 0 \tag{1}$$

belongs to $\mathcal{L}_\infty([0,\infty),\mathbb{R})$.

The *IID problem* is in fact about the noncausal (offline) case, where $\xi(s)$, $s \in [0,\infty)$ is available before the solution η is looked for. If A is stable, then the IID is generated by solving the differential equation (1) directly in forward time. However, such IID cannot be generated for an unstable A. For an unstable A, the basic idea of solving for the IID is to run the stable parts forward in time and the unstable parts backward with the priori information. However, it does not work in the causal (online) case, where only $\xi(s)$, $s \in [0,t]$ is available to determine the solution η at the time t. This problem can be generally formulated as follows:

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‡ $f \in \mathcal{L}_\infty([0,\infty),\mathbb{R}^n)$ denotes that $f(t) \in \mathbb{R}^n$ and $\sup_{t\geq 0}\|f(s)\| < \infty$.

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Causal IID problem: Given $\xi \in \mathcal{L}_\infty([0,T],\mathbb{R})$, $\hat{\eta}_T(0)=0$, $A \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^n$, *and* $\delta > 0$, *find a differentiable function* $\hat{\eta}_T \in \mathcal{L}_\infty([0,T],\mathbb{R}^n)$ *such that* [§]

$$\dot{\hat{\eta}}_T(T) - A\hat{\eta}_T(T) - N\xi(T) \to \mathcal{B}(\delta), \text{ as } T \to \infty.$$

In [7], the noncausal IID problem was considered for a class of forcing terms generated by a known nonlinear exosystem. The problem was further solved for a class of more general systems and a class of more general forcing terms in [10]. However, these inversion-based approaches rely on the availability of the entire output references ahead of time, which is a tough restriction in terms of application. To overcome this limitation, the preview-based stable-inversion approaches were proposed [11, 12]. It requires the finite-previewed (in time) future output reference and thus enables the online implementation. Such a problem can be formulated as a modified *Causal IID problem* that looks for a solution $\hat{\eta}_T \in \mathcal{L}_\infty([0,T],\mathbb{R}^n)$ by $\xi \in \mathcal{L}_\infty\left(\left[0, T+T_{pre}\right],\mathbb{R}\right)$, where $T_{pre} > 0$ is the preview time. It has been shown that a preview time that is long enough is critical to ensure the precision in the preview-based output tracking. However, for some cases, the forcing term $\xi(t)$ in (1) may be an online estimate of uncertainties, which means the future information is unavailable. Therefore, the idea of solving for the noncausal IID is inapplicable to the causal IID problem. To the best of our knowledge, the solutions to the causal IID problem are only limited to a class of bounded forcing term generated by an exosystem. For a class of forcing term generated by a linear exosystem, the IID can be derived precisely by solving a Sylvester equation as proposed in [13]. For a nonlinear exosystem, we have to resort to a first-order partial differential equation as proposed in [14]. The two resulting IID generators can generate the IID directly, which can be considered as static IID generators. However, these two generators require the full knowledge of the state of the exosystem, which may not be obtained directly. Moreover, the resulting IID will preserve the noise if the state of the exosystem is noisy. For these reasons, the authors suppose that a dynamic IID generator was proposed to solve the IID for the (1) in [8]. Furthermore, by using higher order sliding mode differentiators, it was modified in [15] for an unknown matrix A. The proposed methods are efficient and widely used. However, neither dynamic generators treat the system in which A is singular as both of them involve the calculation of A^{-1}. Moreover, the higher order derivatives of the forcing term ξ are required in both methods. This is time-consuming and inaccurate especially in the presence of noise. Furthermore, in the case of a time-varying matrix, they will be time-consuming. For example, if $\frac{d}{dt}A^{-1}(t) = -A^{-1}(t)\frac{d}{dt}A(t)A^{-1}(t)$[¶] is adopted to generate $A^{-1}(t)$ online, then more than n^2 differential equations have to be solved. The same difficulty also exists in solving a time-varying Sylvester equation.

In this paper, we propose a new causal IID generator for a class of perturbed forcing terms generated by linear exosystems. Analysis shows that the (1) is solvable if A is singular under the conditions that are consistent with those for the Sylvester equation proposed in [13]. Furthermore, to suppress the perturbation incurred by the noise, the parameters are partly chosen via $\mathcal{H}_2/\mathcal{H}_\infty$ optimization so that the bound of the error term caused by the perturbation can be calculated. The form of the IID generator and the analysis of its performance are different from those in [8, 15]. The proposed IID generator is also applied to a slowly time-varying unstable differential equation in the simulation section in order to show its effectiveness and superiority. It should be pointed out that the proposed IID generator can also be applied to the tracking problem for nonlinear nonminimum-phase systems by following the similar idea as in [8, 15], that is, to lump weakly nonlinear terms and uncertainties into the forcing term ξ.

The contributions of this paper are as follows, compared with existing similar generators: (i) it circumvents the computation of A^{-1} so that it treats the systems with singular A; (ii) the dimension of the proposed IID generator is lower; (iii) the proposed IID generator avoids taking the higher order derivatives of the forcing term; and iv) it can be easily extended to deal with slowly time-varying matrix differential equations without bringing in extra computational complexity.

[§] $\mathcal{B}(\delta) \triangleq \{\xi \in \mathbb{R} \mid \|\xi\| \le \delta\}$, $\delta > 0$; the notation $x(t) \to \mathcal{B}(\delta)$ means $\inf_{y\in\mathcal{B}(\delta)} |x(t)-y| \to 0$.

[¶] $A(t)A^{-1}(t) = I_n \Rightarrow \frac{d}{dt}A(t)A^{-1}(t) + A(t)\frac{d}{dt}A^{-1}(t) = 0_{n\times n} \Rightarrow \frac{d}{dt}A^{-1}(t) = -A^{-1}(t)\frac{d}{dt}A(t)A^{-1}(t)$

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2. MOTIVATION AND PROBLEM FORMULATION

For the sake of readability, the motivation of the research is given first.

2.1. Motivation

Consider a tracking problem as follows:

$$\begin{aligned} \dot{x}_m &= A_m x_m + B_m u_m, \\ y_m &= C_m x_m, \end{aligned} \tag{2}$$

where $A_m \in \mathbb{R}^{n\times n}$, $x_m, B_m, C_m^{\mathrm{T}} \in \mathbb{R}^n$, $u_m, y_m \in \mathbb{R}$; $C_m B_m$ is nonzero for simplicity. The output y_m is expected to track the desired trajectory $r \in \mathbb{R}$. The signal $r(s)$, $s \in [0, t]$ is available at the time $t > 0$.

Taking the first-order derivative of y_m yields the following:

$$\dot{y}_m = C_m A_m x_m + C_m B_m u_m, \tag{3}$$

which is the external dynamics. Meanwhile, there are ID, written as follows [16, p. 514]:

$$\dot{\eta}_m = A_\eta \eta_m + B_\eta y_m, \tag{4}$$

where $A_\eta \in \mathbb{R}^{(n-1)\times(n-1)}$, $B_\eta, \eta_m \in \mathbb{R}^{n-1}$. There exists a transformation, featured by a square matrix $P \in \mathbb{R}^{n\times n}$, such that

$$x_m = P \begin{bmatrix} \eta_m \\ y_m \end{bmatrix}.$$

In other words, new variables $\begin{bmatrix} \eta_m^{\mathrm{T}} & y_m \end{bmatrix}^{\mathrm{T}}$, which replace x_m, are used to describe the system (2). With respect to the input u_m and output y_m, the system (3)–(4) is equivalent to (2).

If $\max \mathrm{Re}\lambda \left(A_\eta\right) < 0$, the system (2) is *minimum phase*. Then, according to the external dynamics (3), the tracking controller can be designed as follows:

$$u_m = (C_m B_m)^{-1} \left(-C_m A_m x_m + \dot{r} - k\left(y_m - r\right)\right), \tag{5}$$

where $k > 0$. It can ensure $y_m(t) - r(t) \to 0$ as $t \to \infty$. Moreover, according to (4), η_m is bounded because $\max \mathrm{Re}\lambda \left(A_\eta\right) < 0$ and y_m are bounded. On the other hand, the system (2) is *nonminimum phase* if $\max \mathrm{Re}\lambda \left(A_\eta\right) \geq 0$. In this case, η_m will diverge if (5) is still used. Furthermore, u_m will diverge as it contains the component η_m. Therefore, for the nonminimum-phase system, the controller as (5) is infeasible. A solution to such a problem is to identify a *bounded* state reference such as as follows:

$$\dot{\eta}_m^* = A_\eta \eta_m^* + B_\eta r, \tag{6}$$

where η_m^* is bounded (it is not easy as $\max \mathrm{Re}\lambda \left(A_\eta\right) \geq 0$). Consequently, the state reference

$$x_m^* = P \begin{bmatrix} \eta_m^* \\ r \end{bmatrix} \tag{7}$$

is bounded and known. Therefore, according to (3), the ideal feedforward controller is obtained as follows:

$$u_m^* = (C_m B_m)^{-1} \left(\dot{r} - C_m A_m x_m^*\right). \tag{8}$$

Based on (7) and (8), the output tracking problem can be simplified into a stabilization problem. For example, with x_m^* and u_m^* in hand, the reference system of (2) is built as follows:

$$\begin{aligned} \dot{x}_m^* &= A_m x_m^* + B_m u_m^* \\ r &= C_m x_m^*. \end{aligned}$$

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With the reference system previously, an error dynamical system with respect to $x_m - x_m^*$ is further derived. Then, a stabilizing controller in the form of

$$u_m = u_m^* + K\left(x_m - x_m^*\right)$$

is designed, where the gain K is chosen to make $A_m + B_m K$ stable. In this case, the system state x_m is bounded, and the output y_m can track the desired trajectory r. By recalling the whole design process, the key step is to identify the bounded state reference η_m^* in (6). This problem is also considered/concerned with in this paper.

Readers can find many examples of nonminimum-phase systems [2–4, 7, 8]. To make this paper self-contained, a simple example is given to show why nonminimum-phase behavior appears. An unmanned aircraft is treated as a mass point, whose dynamics are given by the following:

$$\begin{aligned} \dot{p} &= v, \\ \dot{v} &= u, \end{aligned} \tag{9}$$

where p, v, and $u \in \mathbb{R}^3$ are the position, velocity, and control signal of the aircraft, respectively. Suppose that a reference velocity $v^* \in \mathbb{R}^3$ is given based on which the produced lift can compensate for the weight of the aircraft. Meanwhile, the position signal of the aircraft needs to be bounded within a certain range so that it can be used for surveillance. If the velocity is considered as the output, then the system (9) is a nonminimum-phase system; the ID of which are $\dot{p} = v$. In order to solve this problem, similar to (6), a key step is to identify a bounded position reference such as

$$\dot{p}^* = v^*,$$

which is a special form of (6). In this case, A_η is not only unstable but also *singular*. This motivates the design of a new generator of causal IID in order to treat the system with singular matrix A.

2.2. *Problem formulation*

Consider the following unstable linear differential equation:

$$\dot{\eta} = A\eta + N\xi, \eta(0) = \eta_0, \tag{10}$$

where $\eta \in \mathbb{R}^n$ is the state and $\xi \in \mathcal{L}_\infty([0,\infty), \mathbb{R})$ (it will be extended to be a vector later) could be modeled as follows:

$$\dot{w} = Sw, \xi = E^{\mathrm{T}} w, \tag{11}$$

where $w \in \mathbb{R}^m, S \in \mathbb{R}^{m\times m}, E \in \mathbb{R}^m$; here, we consider the causal case, namely, the signal $\xi(s)$, $s \in [0, t]$ is available at the time $t > 0$. Moreover, $N \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n\times n}$ is a non-Hurwitz matrix.

Denote $\hat{\eta}$ to be the estimate and define

$$y \triangleq \dot{\hat{\eta}} - A\hat{\eta} - N\xi. \tag{12}$$

The objective is to obtain a bounded estimate $\hat{\eta}$ such that $y(t) \to 0$ as $t \to \infty$. Furthermore, consider the case that ξ is a vector.

Before proceeding further, the following preliminary result is needed.

Lemma 1

If and only if $\mathrm{rank}(F - \lambda I_n) = n - 1$ for every eigenvalue $\lambda \in \mathbb{C}$ of $F \in \mathbb{R}^{n\times n}$, then there exists a vector $B \in \mathbb{R}^n$ such that the pair (F, B) is controllable.

Proof

See Appendix A.1. □

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Remark 1
It should be noted that this useful property was first shown by Wonham [17]. Later, the proof was simplified by Antsaklis [18] in a completely different way. Based on some basic knowledge on matrices, the proof shown in Appendix A.1 is completely different from those in [17, 18]. Moreover, the proof includes the construction of B given in (25). This will be used later in the simulation. Therefore, the proof of the lemma is reserved.

3. A NEW CAUSAL IDEAL INTERNAL DYNAMICS GENERATOR

3.1. *Basic causal ideal internal dynamics generator*

The new causal IID generator is proposed as follows:

$$\dot{x} = A_{cl}x + N_{cl}\xi, x(0) = x_0, \tag{13a}$$
$$\hat{\eta} = C_{cl}^{\mathrm{T}}x, \tag{13b}$$

where

$$x = \begin{bmatrix} v \\ \hat{\eta} \\ e \end{bmatrix} \in \mathbb{R}^{m+n+1}, v \in \mathbb{R}^m, \hat{\eta} \in \mathbb{R}^n, e \in \mathbb{R}, C_{cl} = \begin{bmatrix} 0_{m\times n} \\ I_n \\ 0_{1\times n} \end{bmatrix} \in \mathbb{R}^{(m+n+1)\times n},$$
$$A_{cl} = \begin{bmatrix} S & 0_{m\times n} & L_{11} \\ 0_{n\times m} & A & L_{12} \\ L_{21} & L_{22} & L_3 \end{bmatrix} \in \mathbb{R}^{(m+n+1)\times(m+n+1)}, N_{cl} = \begin{bmatrix} 0_{m\times 1} \\ N \\ 0 \end{bmatrix} \in \mathbb{R}^{m+n+1}, x_0 = \begin{bmatrix} 0 \\ \eta_0 \\ 0 \end{bmatrix}$$
$$L_{11} \in \mathbb{R}^m, L_{12} \in \mathbb{R}^n, L_{21} \in \mathbb{R}^{1\times m}, L_{22} \in \mathbb{R}^{1\times n}, L_3 \in \mathbb{R}.$$

Here, v and e are auxiliary variables. The basic idea is to make (13) satisfied with the following two conditions:

i: $\max \mathrm{Re}\lambda\left(A_{cl}\right) < 0$;
ii: $e(t) \to 0$ as $t \to \infty$.

By taking ξ as the input and x as the state, the condition (i) implies the bounded-input bounded-state stability of (13a), namely, the resulting $\hat{\eta}$ is bounded. On the other hand, (13a) contains the dynamics:

$$\dot{\hat{\eta}} = A\hat{\eta} + L_{12}e + N\xi.$$

According to (12), $y = L_{12}e$. So the condition (ii) implies $y(t) \to 0$ as $t \to \infty$. Therefore, the proposed objective is achieved.

Remark 2
It is easy to make the condition (i) satisfied by choosing appropriate gains $L_{11}, L_{12}, L_{21}, L_{22}$, and L_3. On the other hand, to satisfy the condition (ii), we introduce the dynamics $\dot{v} = Sv + L_{11}e$ into (13), where the matrix S is the same as that in (11). The idea is inspired by a new viewpoint on the internal model principle proposed in [19]: e will vanish if it becomes an input of the internal model such as $\dot{v} = Sv + L_{11}e$, which is further incorporated into a stable closed-loop linear system. These results are stated in Theorems 1–4.

3.2. *Convergence analysis*

Theorem 1
For (13), suppose (i) $\xi \in \mathcal{L}_\infty\left([0,\infty), \mathbb{R}\right)$ is generated by (11); (ii) the gains $L_{11}, L_{12}, L_{21}, L_{22}$, and L_3 satisfy $\max \mathrm{Re}\lambda\left(A_{cl}\right) < 0$. Then $e \to 0$ as $t \to \infty$, meanwhile keeping x bounded. Furthermore, $y = \dot{\hat{\eta}} - A\hat{\eta} - N\xi \to 0$ as $t \to \infty$.

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Proof
See Appendix A.2. □

The essence of Theorem 1 is to find gains $L_{11}, L_{12}, L_{21}, L_{22}$, and$L_3$ satisfying max $\mathrm{Re}\lambda\left(A_{cl}\right) < 0$. However, a question immediately arises as to under what conditions such gains exist for given S and A. This question will be answered in Theorem 2. Denote

$$A_S = \begin{bmatrix} S & 0_{m\times n} \\ 0_{n\times m} & A \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}, L_1 = \begin{bmatrix} L_{11} \\ L_{12} \end{bmatrix} \in \mathbb{R}^{n+m}.$$

Theorem 2
If and only if $\mathrm{rank}(A_S - \lambda I_{n+m}) = n + m - 1$ for every eigenvalue $\lambda \in \mathbb{C}$ of A_S, then there exists a vector $L_1 \in \mathbb{R}^{m+n}$ such that the pair (A_S, L_1) is controllable. Furthermore, if matrices S and A have an eigenvalue in common, then the pair (A_S, L_1) is uncontrollable for any $L_1 \in \mathbb{R}^{m+n}$.

Proof
The first part of Theorem 2 follows from Lemma 1 obviously. If matrices S and A have an eigenvalue in common, denoted by λ_c, then

$$\begin{aligned} \mathrm{rank}\left(A_S - \lambda_c I_{n+m}\right) &= \mathrm{rank}\left(S - \lambda_c I_n\right) + \mathrm{rank}\left(A - \lambda_c I_m\right) \\ &\leq m + n - 2. \end{aligned}$$

We can conclude this proof for the second part of Theorem 2 by Lemma 1. □

With Theorems 1–2 in hand, we have the following:

Theorem 3
For (13), suppose (i) $\xi \in \mathcal{L}_\infty\left([0,\infty), \mathbb{R}\right)$ is generated by (11) with appropriate initial values; (ii) $\mathrm{rank}(A_S - \lambda I) = m + n - 1$ for every eigenvalue λ of A_S. Then (i) there always exist gains $L_{11}, L_{12}, L_{21}, L_{22}$, and$L_3$ satisfying max $\mathrm{Re}\lambda\left(A_{cl}\right) < 0$; furthermore, (ii) $e \to 0$ as $t \to \infty$, meanwhile keeping $x(t)$ bounded. Moreover, $y = \hat{\eta} - A\hat{\eta} - N\xi \to 0$ as $t \to \infty$.

Proof
See Appendix A.3. □

Remark 3
The IID can be given exactly [13]: $\eta = \Pi w$, where $\Pi \in \mathbb{R}^{n\times m}$ satisfies the Sylvester equation $\Pi S = A\Pi + NE^{\mathrm{T}}$. Such equation has a unique solution if and only if S and A have no eigenvalues in common [20, Theorem 13.18, p. 145]. It is easy to see that the following two conditions are equivalent:

$$S \text{ and } A \text{ have no eigenvalues in common} \Leftrightarrow \mathrm{rank}\ \left(A_S - \lambda I_n\right) = n + m - 1.$$

Therefore, the solvability condition of the proposed generator is consistent with that of the Sylvester equation $\Pi S = A\Pi + NE^{\mathrm{T}}$. If A is singular, then S cannot be singular to ensure the existence of the vector $L_1 \in \mathbb{R}^{n+m}$.

Remark 4
The IID can be also given by using the methods in [8, 9]. The forcing ξ can be piecewise modeled by a linear exosystem with known characteristic polynomial: $P(s) = \sum_{i=0}^{m} p_i s^i$, then $\hat{\eta}$ can be generated by the following:

$$\sum_{i=0}^{m} c_i \hat{\eta}^{(i)} = \sum_{i=0}^{m} P_i \xi^{(i)}, \tag{14}$$

where the numbers c_i are chosen to guarantee desirable eigenvalue placement for the convergence and P_i all depend on A^{-1}. For example, $P_0 = c_0 A^{-1} - \left(P_{k-1} + I\right) p_0 A^{-1}$. Unlike such IID

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generator, the proposed IID generator allows A to be singular in some cases. For example, the pair (A_S, L_1) is controllable with

$$A = 0, S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, L_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Moreover, the dimension of the IID generator (14) is $m \times n$, whereas the dimension of the proposed IID generator (13a) is only $m + n + 1$. In addition, the proposed IID generator avoids taking the derivative of ξ. Therefore, the computation complexity is decreased, especially when m and n are large.

Remark 5
The proposed causal IID generator is also applied to (10) with a slowly time-varying matrix $A(t)$. Due to the slowly time-varying matrix $A(t)$, the resulting A_{cl} in (13) is written as $A_{cl}\,(\alpha(t))$, where $\alpha(t) \subset \Gamma \in \mathbb{R}^m$ is continuous and $\|\dot{\alpha}(t)\| \leq \epsilon$ is 'sufficiently' small. For any 'frozen' parameter $\alpha\,(t) \equiv \alpha^*$, system (13) is time invariant, whose equilibrium is denoted by $h\,(\alpha^*, t)$. According to the results mentioned previously,

$$\dot{h}\left(\alpha^*, t\right) - A_{cl}\left(\alpha^*\right) h\left(\alpha^*, t\right) - N_{cl}\xi(t) = 0$$

holds, which further implies $\dot{\hat{\eta}}\,(\alpha^*, t) - A\hat{\eta}\,(\alpha^*, t) - N\xi(t) = 0$. Suppose $\mathrm{Re}\lambda\,(A_{cl}\,(\alpha^*)) < 0$ for any 'frozen' parameter. Then, by the theory of slowly time-varying systems [16, Theorem 9.3, p. 365–368] (some assumptions on $h\,(\alpha^*, t)$ are required), the solution x satisfies that the error $x(t) - h\,(\alpha(t), t)$ is uniformly ultimately bounded with the bound proportional to ϵ. Consequently, $\dot{\hat{\eta}}(t) - A\hat{\eta}(t) - N\xi(t)$ is uniformly ultimately bounded with the bound proportional to ϵ. By using (13) for a slowly time-varying matrix $A(t)$, the proposed IID generator avoids the calculation of $A^{-1}(t)$, which will be time-consuming especially when $A(t)$ is high-dimensional.

3.3. Extension to multiple input

In the following, let us further consider the input ξ as a vector rather than a scalar; that is,

$$\dot{\eta} = A\eta + \sum_{k=1}^{l} N_k \xi_k, \eta(0) = \eta_0, \tag{15}$$

where $\eta \in \mathbb{R}^n, \xi_k \in \mathbb{R}, N_k \in \mathbb{R}^n, k = 1, \ldots, l$. We have the following result:

Theorem 4
For (15), suppose (i) $\xi_k \in \mathcal{L}_\infty\,([0, \infty)\,, \mathbb{R})$ and can be generated by (11) with an appropriate initial value, $k = 1, \ldots, l$; (ii) $\mathrm{rank}(A_S - \lambda I) = m + n - 1$ for every eigenvalue λ of A_S. Then (i) there must exist gains $L_{11}, L_{12}, L_{21}, L_{22}, \text{and} L_3$ satisfying $\max \mathrm{Re}\lambda\,(A_{cl}) < 0$; (ii) furthermore, the following IID generator

$$\begin{aligned} \dot{x} &= A_{cl}x + \sum_{k=1}^{l} N_{cl,k}\xi_k, x\,(0) = x_0 \\ \hat{\eta} &= C_{cl}^{\mathrm{T}} x \end{aligned} \tag{16}$$

can drive $y(t) = \dot{\hat{\eta}}(t) - A\hat{\eta}(t) - \sum_{k=1}^{l} N_k \xi_k\,(t) \to 0$ as $t \to \infty$, meanwhile keeping $x(t)$ bounded, where $x \in \mathbb{R}^{m+n+1}$, $\hat{\eta} \in \mathbb{R}^n$, $N_{cl,k} = \begin{bmatrix} 0_{1\times m} & N_k^{\mathrm{T}} & 0 \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{m+n+1}$, A_{cl}, C_{cl} are the same as in (13).

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Proof
By the additive decomposition [5, 21], the IID generator (16) is decomposed into

$$\begin{aligned} \dot{x}_k &= A_{cl}x_k + N_{cl,k}\xi_k, x_k(0) = x_{0,k} \\ \hat{\eta}_k &= C_{cl}^{\mathrm{T}}x_k, k = 1, \ldots, l \end{aligned} \tag{17}$$

with the relation

$$x = \sum_{k=1}^{l} x_k, \hat{\eta} = \sum_{k=1}^{l} \hat{\eta}_k, x_0 = \sum_{k=1}^{l} x_{0,k}. \tag{18}$$

By conditions (i)–(ii) and *Theorem 3*, the IID generator (17) for each $\xi_k(t)$ can drive $y_k = \dot{\hat{\eta}}_k - A\hat{\eta}_k - N_k\xi_k \to 0$ as $t \to \infty$, meanwhile keeping $x_k(t)$ bounded. By (18), we have the following:

$$y = \dot{\hat{\eta}} - A\hat{\eta} - \sum_{k=1}^{l} N_k\xi_k = \sum_{k=1}^{l}\left(\dot{\hat{\eta}}_k - A\hat{\eta}_k - N_k\xi_k\right) \to 0$$

as $t \to \infty$, meanwhile keeping $x(t) = \sum_{k=1}^{l} x_k(t)$ bounded. □

4. $\mathcal{H}_2/\mathcal{H}_\infty$ OPTIMAL DESIGN OF IDEAL INTERNAL DYNAMICS GENERATOR

So far, we have proposed the structure of the IID generators and further investigated the existence of their parameters $L_{11}, L_{12}, L_{21}, L_{22}, \text{and} L_3$. However, there exist an infinite number of choices of the parameters $L_{11}, L_{12}, L_{21}, L_{22}, \text{and} L_3$ to satisfy $\max \mathrm{Re}\lambda\left(A_{cl}\right) < 0$. In this section, we will design these parameters according to some optimization principles.

In practice, the forcing term ξ cannot always be modeled as (11) without perturbation. Assume $\varepsilon \in \mathbb{R}$ to be a bounded perturbation. Driven by $\xi + \varepsilon$, the solution to (13) satisfies the following:

$$\begin{aligned} \dot{x}_\varepsilon &= A_{cl}x_\varepsilon + N_{cl}\left(\xi + \varepsilon\right), x_\varepsilon(0) = x_0, \\ \hat{\eta}_\varepsilon &= C_{cl}^{\mathrm{T}}x_\varepsilon. \end{aligned} \tag{19}$$

It is expected that $L_{11}, L_{12}, L_{21}, L_{22}, \text{and} L_3$ be designed such that $\hat{\eta}_\varepsilon - \hat{\eta}$ is not sensitive to the perturbation ε. Subtracting (13) from (19) results in the following:

$$\begin{aligned} \dot{x}_e &= A_{cl}x_e + N_{cl}\varepsilon, x_e(0) = 0, \\ \hat{\eta}_e &= C_{cl}^{\mathrm{T}}x_e, \end{aligned} \tag{20}$$

where $\hat{\eta}_e = \hat{\eta}_\varepsilon - \hat{\eta}$ and $x_e = x_\varepsilon - x$. Denote

$$A'_S = \begin{bmatrix} A_S & L_1 \\ 0_{1\times(m+n)} & 0 \end{bmatrix}, L_{23} = \begin{bmatrix} L_{21} & L_{22} & L_3 \end{bmatrix}^{\mathrm{T}}.$$

Then $A_{cl} = A'_S + B_1L_{23}^{\mathrm{T}}$. The system (20) can be rewritten as follows:

$$\begin{aligned} \dot{x}_e &= A'_S x_e + B_1 u + N_{cl}\varepsilon, x_e(0) = 0, \\ \hat{\eta}_e &= C_{cl}^{\mathrm{T}}x_e, \\ u &= L_{23}^{\mathrm{T}}x_e, \end{aligned}$$

which is shown in Figure 1.

Although the $\mathcal{H}_\infty$ criterion ensures some tracking performance as well as robustness, the insensitivity against noise is more naturally enforced by the $\mathcal{H}_2$ criterion. Robust pole placement specifications are also required for reasonable feedback gains. Denote by $T_{\hat{\eta}_e\varepsilon}$ the closed-loop transfer functions from ε to $\hat{\eta}_e$. For simplicity, we determine L_1 similar to (25) beforehand. Then, our goal is to design a state-feedback law $u = L_{23}^{\mathrm{T}}x_e$ that

- Maintains $\left\|T_{\hat{\eta}_e\varepsilon}\right\|_\infty$ below some prescribed value $\gamma_0 > 0$.

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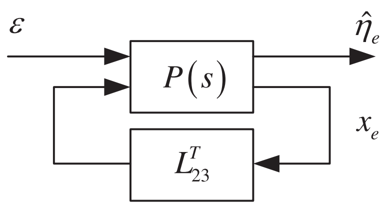

Figure 1. State-feedback control.

- Maintains $\|T_{\hat{\eta}_e\varepsilon}\|_2$ below some prescribed value $\nu_0 > 0$.
- Minimizes an $\mathcal{H}_2/\mathcal{H}_\infty$ trade-off criterion of the form $\alpha\|T_{\hat{\eta}_e\varepsilon}\|_\infty + \beta\|T_{\hat{\eta}_e\varepsilon}\|_2$, $\alpha \geq 0, \beta \geq 0$.
- Places the closed-loop poles in a prescribed region $\mathcal{D}$ of the open left-half plane.

Formally, the objective is to find L_{23} such that

$$\begin{array}{ll} \min\limits_{L_{23}} & \alpha\|T_{\hat{\eta}_e\varepsilon}\|_\infty + \beta\|T_{\hat{\eta}_e\varepsilon}\|_2 \\ \text{s.t.} & \|T_{\hat{\eta}_e\varepsilon}\|_\infty < \gamma_0 \\ & \|T_{\hat{\eta}_e\varepsilon}\|_2 < \nu_0 \\ & \lambda(A_{cl}) \in \mathcal{D} = \{z \in \mathbb{C} \mid Q + Mz + M\bar{z} < 0\} \end{array}, \tag{21}$$

where $Q = Q^{\mathrm{T}}$ and matrix M is properly chosen.

Remark 6
It is not necessary that the perturbation $\varepsilon \in \mathcal{L}_2$ in practice although $\mathcal{H}_2$ optimization is considered. From (13), the state is still bounded if ε is bounded and $\max \operatorname{Re}\lambda(A_{cl}) < 0$.

Remark 7
The MATLAB function 'msfsyn' [22] is applicable to solve the optimization problem (21).

Remark 8
The parameter L_{23} of the proposed IID generator is chosen via $\mathcal{H}_2/\mathcal{H}_\infty$ optimization (21). However, a solution may not be found if γ_0, ν_0 are small or inappropriate. Alternatively, the optimization can be modified as follows:

$$\begin{array}{ll} \min\limits_{L_{23}} & \alpha\|T_{\hat{\eta}_e\varepsilon}\|_\infty + \beta\|T_{\hat{\eta}_e\varepsilon}\|_2 \\ \text{s.t.} & \lambda(A_{cl}) \in \mathcal{D} = \{z \in \mathbb{C} \mid Q + Mz + M\bar{z} < 0\}. \end{array}$$

If the pair (A'_S, B_1) is controllable, then a solution can always be found.

5. SIMULATION EXAMPLES

For simplicity, in the following examples, the prescribed region $\mathcal{D}$ of the open left-half plane is chosen to be an intersection of a conic sector centered at the origin with inner angle $\frac{3\pi}{4}$ and a vertical strip $[-10, -1]$, shown in Figure 2.

Example 1
In (10), $A = 0, N = 1$, where ξ is generated by (11) with $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $w(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Because A_S has three different eigenvalues $0, \pm j$, $\operatorname{rank}(A_S - \lambda I) = 2$ for $\lambda = 0, \pm j$. Because A is nonsingular, the dynamic IID generators proposed in [8, 9] are inapplicable to this example. Similar to (25) in Appendix A.1, L_1 is chosen as $L_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$. Choosing $\gamma_0 = 20, \nu_0 = 20, \alpha = 0.5, \beta = 0.5$ and solving (21) by the MATLAB function 'msfsyn', we obtain $L_{23} = 10^3 \times \begin{bmatrix} 0.5360 & 1.0746 & -0.9743 & -0.0219 \end{bmatrix}^{\mathrm{T}}$ with $\|T_{\hat{\eta}_e\varepsilon}\|_\infty = 1.75$ and $\|T_{\hat{\eta}_e\varepsilon}\|_2 = 2.61$. By solving (13) in forward time, the IID is obtained. As shown in Figure 3, the one-dimensional estimated IID $\hat{\eta}$ is bounded and $y = \dot{\hat{\eta}} - A\hat{\eta} - N\xi \to 0$ as $t \to \infty$. Moreover, it is easy to see that the estimated

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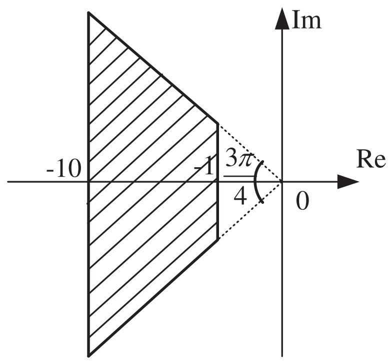

Figure 2. Prescribed region $\mathcal{D}$.

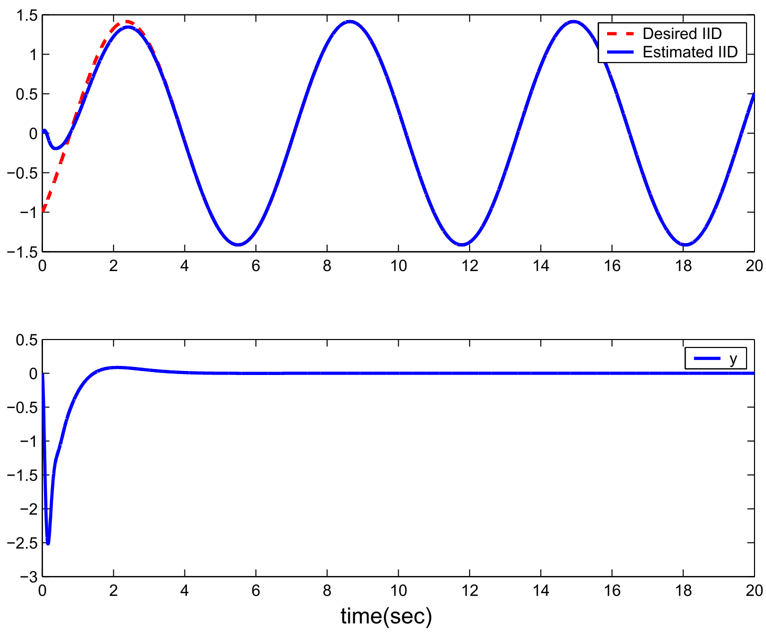

Figure 3. Output of the ideal internal dynamics (IID) generator in Example 1. [Colour figure can be viewed at wileyonlinelibrary.com]

IID $\hat{\eta}$ converges to the desired IID. In the presence of ε, as shown in Figure 4, it is easily observed that the estimated IID can also converge to the desired IID with a small error.

Example 2
In (15),

$$A(t) = \begin{bmatrix} 0.2\sin(0.05t) & 1 \\ -1 & 1 \end{bmatrix}, N_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, N_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, l = 2, \tag{22}$$

where ξ_1, ξ_2 are generated respectively by (11) with

$$S = \begin{bmatrix} 0 & 0.2 \\ -0.2 & 0 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, w_1(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^{\mathrm{T}}, w_2(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^{\mathrm{T}}.$$

In the IID generator (16), A will be replaced by $A(t)$ in (22) to obtain an approximate IID. However, in order to design L_1 and L_{23}, we consider $A(t) \equiv \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ first. Similar to (25) in Appendix A.1, L_1 is designed as $L_1 = \begin{bmatrix} 1 & 0 & 1 & 0.0670 \end{bmatrix}^{\mathrm{T}}$. Choosing $\gamma_0 = 20, \nu_0 =$

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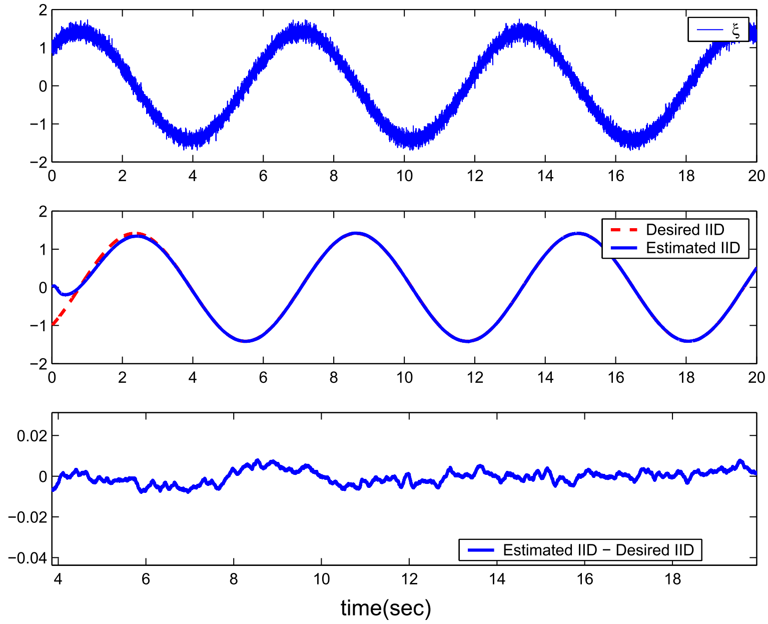

Figure 4. Output of the ideal internal dynamics (IID) generator in Example 1 in the presence of noise. [Colour figure can be viewed at wileyonlinelibrary.com]

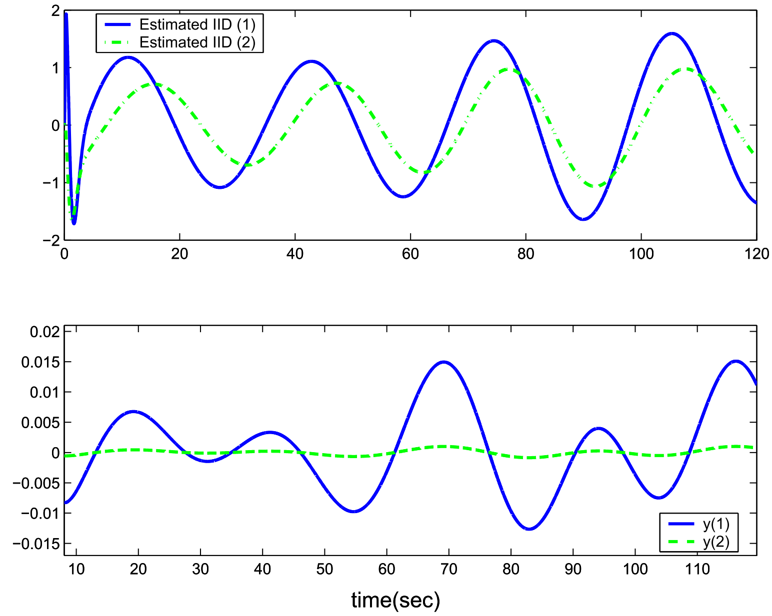

Figure 5. Output of the ideal internal dynamics (IID) generator in Example 2 with a time-varying matrix. [Colour figure can be viewed at wileyonlinelibrary.com]

$20, \alpha = 0.5, \beta = 0.5$ and solving (21) by the MATLAB function 'msfsyn', we obtain $L_{23} = 10^4 \times \begin{bmatrix} -0.5702 & 1.0009 & 0.5159 & 0.0850 & -0.0025 \end{bmatrix}^{\mathrm{T}}$ with $\left\| T_{\hat{\eta}_e \varepsilon} \right\|_\infty = 9.37$ and $\left\| T_{\hat{\eta}_e \varepsilon} \right\|_2 = 16$. By solving the resulting IID generator (16) in forward time, the estimated IID is obtained. As shown in Figure 5, the two-dimensional estimated IID $\hat{\eta}$ is bounded, and each element of $y = \dot{\hat{\eta}} - A\hat{\eta} - N\xi \in \mathbb{R}^2$ is bounded ultimately by a very small positive value.

Example 3

In (10), $A = 2, N = 1, S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, w(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, w(10) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$. The forcing term ξ is a continuous signal, namely,

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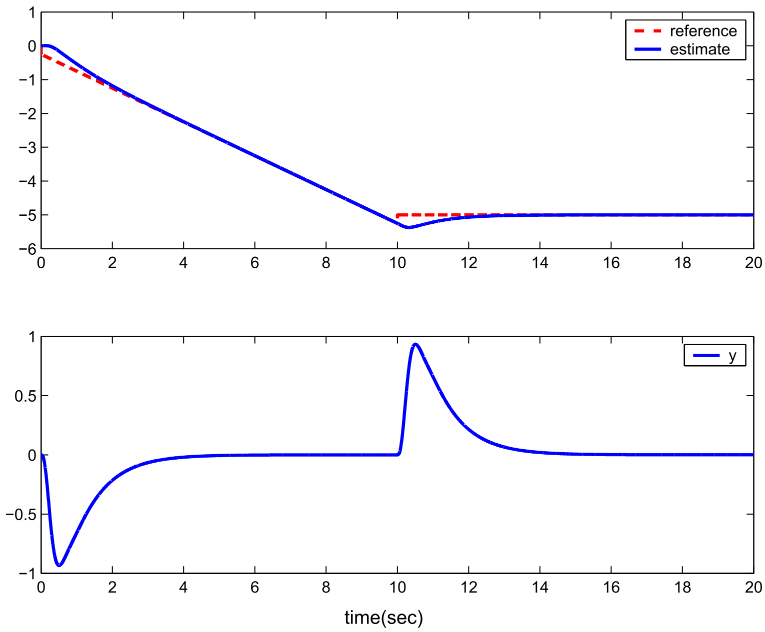

Figure 6. Output of the ideal internal dynamics generator in Example 3. [Colour figure can be viewed at wileyonlinelibrary.com]

$$\xi(t) = \begin{cases} t & t \in [0, 10] \\ 10 & t \in (10, 20] \end{cases},$$

which can be piecewise modeled by (11) with the given parameters. Because A_S has three different eigenvalues $0, \pm\sqrt{2}j$, $\text{rank}(A_S - \lambda I) = 2$ for every eigenvalue λ. Similar to (25) in Appendix A.1, L_1 is chosen as $L_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Choosing $\gamma_0 = 20, \nu_0 = 20, \alpha = 0.5$, and$\beta = 0.5$ and solving (21) by the MATLAB function 'msfsyn', we can obtain $L_{23} = \begin{bmatrix} 73.44 & 129.40 & -245.56 & -15.63 \end{bmatrix}^T$. As shown in Figure 6, it is easily observed that the estimate converges to the reference with a nice transient behavior.

Remark 9
In the simulation, the transient performance is not always satisfied, because the number of parameters used to change matrix A_{cl} is less than its dimension. As a result, there is less degree of freedom to adjust the zeros of the system or to find a solution to the optimization problem (21). It is noted that the auxiliary variable e is only one-dimensional in this paper. So a solution is to introduce more auxiliary variables. However, if more auxiliary variables are used, the proof of Theorem 1 has to be modified.

6. CONCLUSIONS

In this paper, a new dynamic generator for causal IID is proposed. By solving the problem in forward time, the IID can be obtained. Owing to the dynamics, it can suppress noise and perturbations. Compared with existing generators, it is applicable to systems with a singular A, incurs less computation burden, and avoids taking the higher order derivatives of the forcing term. Furthermore, the proposed generator can be easily extended to deal with slowly time-varying unstable linear differential equations in the same framework without extra computation. The simulation examples demonstrate the effectiveness of the proposed IID generator.

APPENDIX

A.1. Proof of Lemma 1

Before presenting the proof, we introduce a lemma.

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Lemma 2 (Popov-Belevitch-Hautus (PBH) controllability test [23], *Theorem 4.8, p.102)*
The matrix pair (F, B) is controllable if and only if

$$\mathrm{rank}\left[\, F - \lambda I \;\; B \,\right] = n$$

for every eigenvalue $\lambda \in \mathbb{C}$ of F.

Sufficiency of Lemma 1 (a constructive proof)
For $F \in \mathbb{R}^{n\times n}$, there exists a matrix $T \in \mathbb{R}^{n\times n}$ such that [20, Theorem 9.22, p.82–83]

$$T^{-1}FT = J = \mathrm{diag}\left(J_1, \ldots, J_{n_s}\right),$$

where each of the Jordan block matrices $J_1, \ldots, J_{n_s}$ is of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \tag{23}$$

in the case of real eigenvalues λ_i, and

$$J_i = \begin{bmatrix} M_i & I_2 & & \\ & M_i & \ddots & \\ & & \ddots & I_2 \\ & & & M_i \end{bmatrix}, \tag{24}$$

where $M_i = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in the case of $\alpha_i \pm j\beta_i, \beta_i \neq 0$. For simplicity and without loss of generality, we assume that only the last Jordan block J_{n_s} is in the form of (24). The Jordan block J_i corresponds to a left eigenvector $v_i \in \mathbb{R}^n, i = 1, \ldots, n_s - 1$, and J_{n_s} corresponds to a couple of left eigenvectors $v_{n_s} \in \mathbb{C}^n, v_{n_s+1} \in \mathbb{C}^n$. It is easy to see that $v_i^{\mathrm{H}} v_k = 0$ by the form of J, $i \neq k$, except for v_{n_s} and v_{n_s+1}. Every eigenvalue $\lambda_i \in \mathbb{C}$ corresponds to a left eigenvector $0 \neq v_i \in \mathbb{C}^n$ such that $v_i^{\mathrm{H}} J = \lambda_i v_i^{\mathrm{H}}, i = 1, \ldots, n_s+1$, which implies that $\bar{v}^{\mathrm{H}} J = \bar{\lambda}\bar{v}^{\mathrm{H}}$. Here, $\bar{x}$ represents the element-by-element conjugation of $x \in \mathbb{C}^n$, and x^{H} represents the conjugate transpose of $x \in \mathbb{C}^n$. Therefore, for a couple of conjugate complex roots, eigenvectors associated with them can be chosen to be conjugate; that is, $v_{n_s+1} = \bar{v}_{n_s}$, so that vector

$$B = T \sum_{i=1}^{n_s+1} v_i \tag{25}$$

is real. Next, we will show that $\mathrm{rank}\left[\, F - \lambda I_n \;\; B \,\right] = n$ for every eigenvalue $\lambda \in \mathbb{C}$ of F. Suppose, to the contrary, that there exists a vector $0 \neq p \in \mathbb{C}^n$ and $\lambda_k \in \mathbb{C}$ such that $p^{\mathrm{H}}\left[\, F - \lambda_k I_n \;\; B \,\right] = 0$; that is,

$$\begin{aligned} p^{\mathrm{H}}\left(F - \lambda_k I_n\right) &= 0, \\ p^{\mathrm{H}} B &= 0. \end{aligned}$$

Furthermore, we have the following:

$$p^{\mathrm{H}} T \left(J - \lambda_k I_n\right) = 0. \tag{26}$$

$$p^{\mathrm{H}} T \sum_{i=1}^{n_s+1} v_i = 0. \tag{27}$$

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Because rank$(F - \lambda I_n)$ =rank$(J - \lambda I_n) = n-1$ for every eigenvalue of F, each eigenvalue corresponds to exactly one eigenvector. As a result, (26) implies $T^{\mathrm{H}} p = \mu v_k, 0 \neq \mu \in \mathbb{C}$. Furthermore, the (27) implies

$$\mu v_k^{\mathrm{H}} v_k = 0, k = 1, \ldots, n_s - 1, \tag{28}$$

or

$$\mu v_{n_s}^{\mathrm{H}} \left(v_{n_s} + \bar{v}_{n_s}\right) = 0, k = n_s, \tag{29}$$

or

$$\mu \bar{v}_{n_s}^{\mathrm{H}} \left(v_{n_s} + \bar{v}_{n_s}\right) = 0, k = n_s + 1, \tag{30}$$

where the orthogonality and $v_{n_s+1} = \bar{v}_{n_s}$ have been utilized. The (28) implies that

$$v_k = 0,$$

which contradicts with $v_i \neq 0_n, i = 1, \ldots, n_s + 1$. Equation (29) or (30) implies that

$$v_{n_s} + \bar{v}_{n_s} = 0.$$

Consequently, v_{n_s} is in the form of $v_{n_s} = j v_{n_s}^r$, where $v_{n_s}^r \in \mathbb{R}^n$. Because

$$\begin{aligned} v_{n_s}^{\mathrm{H}} J &= \left(\alpha_{n_s} + j\beta_{n_s}\right) v_{n_s}^{\mathrm{H}}, \\ \bar{v}_{n_s}^{\mathrm{H}} J &= \left(\alpha_{n_s} - j\beta_{n_s}\right) \bar{v}_{n_s}^{\mathrm{H}}, \end{aligned}$$

we have the following:

$$\begin{aligned} 0 &= \left(v_{n_s}^{\mathrm{H}} + \bar{v}_{n_s}^{\mathrm{H}}\right) J \\ &= \left(\alpha_{n_s} + j\beta_{n_s}\right) v_{n_s}^{\mathrm{H}} + \left(\alpha_{n_s} - j\beta_{n_s}\right) \bar{v}_{n_s}^{\mathrm{H}}, \\ &= 2\beta_{n_s} v_{n_s}^r j \end{aligned}$$

which contradicts with $\beta_{n_s} \neq 0$ or $v_{n_s} \neq 0_n$. Therefore, rank$\left[\, F - \lambda I_n \;\; B \,\right] = n$ for every eigenvalue $\lambda \in \mathbb{C}$ of F, namely, the pair (F, B) is controllable by Lemma 2. □

Necessity of Lemma 1

If rank$(F - \lambda I_n) \neq n - 1$, namely, rank$(F - \lambda I_n) \leq n - 2$ for every eigenvalue λ of F, then rank$\left[\, F - \lambda I_n \;\; B \,\right] \leq n-1$ for any $B \in \mathbb{R}^n$, namely, the pair (F, B) is uncontrollable by Lemma 2. □

A.2. Proof of Theorem 1

Before presenting the proof, we introduce a lemma.

Lemma 3

If the pair (F, B) is controllable, then there exists a vector $C \in \mathbb{R}^n$ such that

$$C^{\mathrm{T}} (s I_n - F)^{-1} B = \frac{1}{\det\left(s I_n - F\right),}$$

where $F \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^n$.

Proof

First, we have $(s I_n - F)^{-1} B = G \left[\, s^{n-1} \; \ldots \; 1 \,\right]^{\mathrm{T}} / \det\left(s I_n - F\right)$, where $G \in \mathbb{R}^{n \times n}$. If the pair (F, B) is controllable, then the matrix G is of full rank [24]. We can complete this proof by choosing $C = \left(G^{-1}\right)^{\mathrm{T}} \left[\, 0 \; \ldots \; 0 \;\; 1 \,\right]^{\mathrm{T}}$. □

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Proof of Theorem 1
The IID generator (13) contains the dynamics $\dot{v} = Sv + L_{11}e$. Its Laplace transformation is as follows:

$$v(s) = (sI_m - S)^{-1} L_{11}e(s).$$

The condition $\max \mathrm{Re}\lambda (A_{cl}) < 0$ implies that the pair (S, L_{11}) is controllable. Further, by Lemma 3, there exists a vector $C_e \in \mathbb{R}^m$ such that

$$C_e^{\mathrm{T}} v(s) = C_e^{\mathrm{T}} (sI_m - S)^{-1} L_{11}e(s) = \frac{1}{\det (sI_m - S)} e(s),$$

namely,

$$e(s) = \det (sI_m - S)\, C_e^{\mathrm{T}} v(s). \tag{31}$$

By (13), the transfer function from ξ to v is as follows:

$$v(s) = C_v^{\mathrm{T}} (sI_{m+n+1} - A_{cl})^{-1} N_{cl}\xi(s),$$

where $C_v = \begin{bmatrix} I_m & 0_{m\times n} & 0 \end{bmatrix}^{\mathrm{T}}$. Substituting the equation previously into (31) yields the following:

$$e(s) = \det (sI_m - S)\, C_e^{\mathrm{T}} C_v^{\mathrm{T}} (sI_{m+n+1} - A_{cl})^{-1} N_{cl}\xi(s).$$

Because ξ is generated by (11), we have $\xi(s) = E^{\mathrm{T}} (sI_m - S)^{-1} w(0)$, where $w(0) \in \mathbb{R}^m$. Because $(sI_m - S)^{-1} = \frac{1}{\det(sI_m - S)} \mathrm{adj}(sI_m - S)$, $e(s)$ is further represented as follows:

$$\begin{aligned} e(s) &= \det (sI_m - S)\, C_e^{\mathrm{T}} C_v^{\mathrm{T}} (sI_{m+n+1} - A_{cl})^{-1} N_{cl} E^{\mathrm{T}} \frac{1}{\det (sI_m - S)} \mathrm{adj}\, (sI_m - S)\, w(0) \\ &= C_e^{\mathrm{T}} C_v^{\mathrm{T}} (sI_{m+n+1} - A_{cl})^{-1} N_{cl} E^{\mathrm{T}} \mathrm{adj}\, (sI_m - S)\, w(0). \end{aligned} \tag{32}$$

Because $\max \mathrm{Re}\lambda (A_{cl}) < 0$ and the order of A_{cl} is higher than that of S, for any initial values $w(0)$, we have $e \to 0$ as $t \to \infty$ from (32). Because ξ is bounded on $[0, \infty)$ and $\max \mathrm{Re}\lambda (A_{cl}) < 0$, the signals v and $\hat{\eta}$ in (13) are bounded. Because the IID (13) contains the relation $\dot{\hat{\eta}} = A\hat{\eta} + L_{12}e + N\xi$. Considering the obtained result that $e \to 0$ as $t \to \infty$, we have $y = L_{12}e = \dot{\hat{\eta}} - A\hat{\eta} - N\xi \to 0$ as $t \to \infty$. □

A.3. *Proof of Theorem 3*

By condition (ii) and Theorem 2, there exists a vector L_1 such that the pair (A_S, L_1) is controllable. Consider the pair:

$$\left(\begin{pmatrix} A_S & L_1 \\ 0_{1\times(n+m)} & 0 \end{pmatrix}, \begin{pmatrix} 0_{n+m} \\ 1 \end{pmatrix} \right). \tag{33}$$

The resulting controllability matrix is as follows:

$$W = \begin{pmatrix} 0_{n+m} & L_1 & A_S L_1 & \cdots & A_S^{n+m-1} L_1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Because the pair (A_S, L_1) is controllable, $\mathrm{rank}\begin{pmatrix} 0_{n+m} & L_1 & A_S L_1 & \cdots & A_S^{n+m-1} L_1 \end{pmatrix} = n + m$. Consequently, $\mathrm{rank} W = n + m + 1$. Therefore, the pair (33) is controllable, namely, there always exist gains $L_{21}, L_{22}, \mathrm{and} L_3$ such that $\max \mathrm{Re}\lambda (A_{cl}) < 0$. The remainder of proof is similar as Theorem 1.

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