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Saturated repetitive control for a class of nonlinear systems: A contraction mapping method

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A B S T R A C T

Contraction mapping methods do not need to know about the concrete form of plant models like the Lyapunov method. This is the biggest advantage over other methods in the field of iterative learning control for example. However, it is difficult to use such a tool to analyze repetitive control systems. This paper proposes a contraction mapping method based on spectral theory to design a saturated repetitive controller for a class of nonlinear systems, where the derived *necessary and sufficient* condition on the spectral radius can reduce the conservatism as much as possible. The feasibility of our work is demonstrated through a robotic manipulator tracking example.

1. Introduction

Repetitive control (RC, or repetitive controller, also designated RC) and iterative learning control (ILC, or iterative learning controller, also designated ILC) are two closely related control techniques. The differences depend on whether the trajectory is repeated over $[0, \infty)$ or a finite interval, as well as the requirement for initial condition resetting. The analysis methods used are very different between RC and ILC in most literature. Design methods for RC are mainly based on the internal model principle and frequency analysis techniques [1], whereas the contraction mapping methods are often applied to designing ILC.

Owing to the difference mentioned above, most RC schemes are limited to linear systems. However, the nonlinear RC problems are receiving more and more attention in recent years, where the leading method is the adaptive-control-like method [2]. A novel learning approach was described in [3] for asymptotic state tracking in a class of nonlinear systems. Compared with the previous methods, the main advantage of the proposed learning approach was computationally simple and did not require solving any complicated equations based on full system dynamics. Hybrid control schemes were developed in [4], which utilized the RC term to compensate for periodic dynamics and other methods to compensate for aperiodic dynamics. An adaptive RC was proposed for a class of nonlinearly parameterized systems in [5]. Both partially and fully saturated learning laws were analyzed in detail and compared. A further result of a class of periodic time varying nonlinear systems was presented in [6]. By considering that many RC schemes required plants to be parameterizable, an RC was integrated with robust adaptive control in [7] by using the backstepping design for a class of cascade systems without parametrization. A continuous-time RC was proposed in [8] to track periodic trajectories for a class of nonlinear dynamical systems with nonparametric uncertainty and unknown state-dependent control direction matrix. In order to achieve a tradeoff between tracking performance and stability, a filtered RC (FRC) was proposed for a class of nonlinear systems in [9]. Moreover, the proposed FRC can deal with small input delay while the corresponding RC cannot. A novel learning control in [10] was designed on the basis of the ideas presented in [4]. Not only global asymptotic tracking was achieved but also sufficient conditions for the asymptotic "input learning" were derived. A control law with finite memory was also designed. The classical PID$^{\rho-1}$ control combined with RC was used for the output regulation of a class of minimum-phase, nonlinear systems with unknown output-dependent nonlinearities, unknown parameters and known relative degree [11]. A further result was presented in [12] to extend the nonlinear systems considered in [11]. Padé approximates were used in the implementation of RCs to solve the output tracking problem (via output error feedback) in the presence of uncertain periodic reference and/or disturbance signals with known common period [13].

These control schemes all utilize Lyapunov-based design techniques to develop feedforward to compensate for unknown disturbances. Moreover, the designed controllers depend on the concrete forms of the Lyapunov functions. Indeed, the Lyapunov-based design technique is an excellent choice for nonlinear RC problems. However, in most cases, there is still no general method to construct Lyapunov functions. By taking this into account, contraction

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mapping methods are expected to apply to RC. Such a design does not need to know about the concrete form of plant models like the Lyapunov method. This is the biggest advantage over other methods. However, it is difficult to use this tool in designing RC, because of the initial condition not resetting anymore. A formalism of ILC has been used in [14] to solve an RC problem for forcing a system to track a prescribed periodic reference signal. The proposed method adopts the idea of contraction mapping. However, the proposed method is only applicable to discrete-time systems. Moreover, it cannot be applied to the rejection of periodic disturbances. Similarly, based on the contraction mapping, a conditional learning control was proposed to track periodic signals for a class of nonlinear systems with unknown dynamics [15]. The learning is based on the steady-state output so that the updating law works only if a particular condition is satisfied. With such a mechanism, monotonic convergence of the control sequence in the iteration domain is achieved. The price to pay is the increase in the number of repetitive trials.

In this paper, a contraction mapping method is proposed to design a saturated D-type RC for a class of nonlinear systems, which is updated at every trial. By the converse Lyapunov theorem, an operator mapping is established between the input error and the state error of two successive periods. In order to reduce the conservatism, a necessary and sufficient condition is derived for the contraction mapping. Compared with existing studies, the main contributions of this paper are: (i) a new RC scheme subject to actuator saturation is proposed for a class of nonlinear systems without resorting to design Lyapunov functions; (ii) nonlinear RC systems are analyzed by contraction mapping and spectral theory with a derived necessary and sufficient condition so that the conservatism is reduced as much as possible.

The following notation is used. $\mathbb{R}^n$ is the Euclidean space of dimension n and $\mathbb{R}_+$ denotes the space of nonnegative real in $\mathbb{R}$. $\|\cdot\|$ denotes the Euclidean norm or a matrix norm induced by the Euclidean norm. $\mathcal{C}([a,b],\mathbb{R}^n)$ denotes the space of continuous n-dimensional vector functions on $[a,b]$. $\mathcal{C}^m_{PT}([0,\infty),\mathbb{R}^n)$ is the space of mth-order continuously differentiable functions $\mathbf{f}:[0,\infty)\to\mathbb{R}^n$ which are T-periodic, i.e. $\mathbf{f}(t+T)=\mathbf{f}(t)$. $\|\mathbf{x}\|_{[0,T]}\triangleq\sup_{t\in[0,T]}\|\mathbf{x}(t)\|$, where $\mathbf{x}\in\mathcal{C}([0,T],\mathbb{R}^n)$. Let $\mathcal{A}$ be a linear compact operator with $\sigma(\mathcal{A})$ the spectrum and $r_{\mathcal{A}}=\sup_{z\in\sigma(\mathcal{A})}|z|$ the spectral radius. $\mathbf{I}_n$ is an identity matrix with dimension n. $\mathcal{D}^+\mathbf{x}$ denotes the upper right Dini derivative of a function and is defined by $\mathcal{D}^+\mathbf{x}(t_0)\triangleq\begin{bmatrix}\lim_{t\to t_0+0}\sup\frac{x_1(t)-x_1(t_0)}{t-t_0} & \cdots\\ \lim_{t\to t_0+0}\sup\frac{x_n(t)-x_n(t_0)}{t-t_0}\end{bmatrix}^{\mathrm{T}}$, where $\mathbf{g}(t)\triangleq\begin{bmatrix}g_1(t) & \cdots & g_n(t)\end{bmatrix}^{\mathrm{T}}\in\mathbb{R}^n$.

2. Preliminary: Contraction mapping

Define $\xi_{i+1}(\tau)\triangleq\xi_i(\tau+T)$, where $\xi=x,y$ and $x_i,y_i:[0,T]\to\mathbb{R}_+$. They have the following relationship

$$x_i(\tau)\le e^{-a_1\tau}x_i(0)+a_2\int_0^\tau e^{-a_1(\tau-s)}y_i(s)\,\mathrm{d}s \tag{1}$$

$$y_{i+1}(\tau)\le a_3y_i(\tau)+a_4x_i(\tau) \tag{2}$$

where $a_1,a_2,a_3,a_4\in\mathbb{R}_+$, $i=0,1,\ldots$. Rewrite inequalities (1) and (2) as follows:

$$x_{i+1}(0)\le e^{-a_1T}x_i(0)+a_2\int_0^T e^{-a_1(T-s)}y_i(s)\,\mathrm{d}s$$

$$y_{i+1}(\tau)\le a_4e^{-a_1\tau}x_i(0)+a_3y_i(\tau)+a_2a_4\int_0^\tau e^{-a_1(\tau-s)}y_i(s)\,\mathrm{d}s.$$

Define the following operators as

$$\mathcal{Q}_0(u)\triangleq a_2\int_0^T e^{-a_1(T-s)}u(s)\,\mathrm{d}s,\ u\in\mathcal{C}([0,T],\mathbb{R})$$

$$\mathcal{Q}_1(u)\triangleq a_2\int_0^\tau e^{-a_1(\tau-s)}u(s)\,\mathrm{d}s,\ u\in\mathcal{C}([0,T],\mathbb{R})$$

$$\mathcal{Q}_2(u)(\tau)\triangleq e^{-a_1\tau}u,\ u\in\mathbb{R}_+$$

and

$$\mathbf{z}_i=\begin{pmatrix}x_i(0)\\ y_i\end{pmatrix}\in\mathbb{R}\times\mathcal{C}([0,T],\mathbb{R}).$$

Then inequalities (1) and (2) can be written in a compact form as

$$\mathbf{z}_{i+1}\le\mathcal{Q}\mathbf{z}_i \tag{3}$$

where

$$\mathcal{Q}=\begin{pmatrix}e^{-a_1T} & \mathcal{Q}_0\\ a_4\mathcal{Q}_2 & a_3+a_4\mathcal{Q}_1\end{pmatrix}. \tag{4}$$

It is easy to verify that $\mathbb{R}\times\mathcal{C}([0,T],\mathbb{R})$ is a Banach space and $\mathcal{Q}$ is a compact linear map.

Lemma 1. *If and only if*

$$a_3<1 \text{ and } \frac{a_2a_4}{a_1(1-a_3)}<1 \tag{5}$$

then $r_{\mathcal{Q}}<1$, where $\mathcal{Q}$ is defined in (4).

Proof. This proof is divided into three steps, namely *condition conversion, proof of sufficiency,* and *proof of necessity*, where the condition conversion serves for the following two steps. Refer to the Appendix for details. □

Lemma 2. *For (1) and (2), if (5) holds, then $\|y_i\|_{[0,T]}$, $|x_i(0)|$ converge to zero exponentially.*

Proof. According to Lemma 1, $r_{\mathcal{Q}}<1$ if and only if (5) holds. Furthermore, from [16], $r_{\mathcal{Q}}<1$ if and only if $\left\|\mathcal{Q}^i\right\|_B<Me^{-\omega i}$ for some $M,\omega\in\mathbb{R}_+$, where $\|\cdot\|_B$ is induced by $\|\cdot\|_c$ [17, section 15.1]. Let $\bar{\mathbf{z}}_i=\mathcal{Q}^i\mathbf{z}_0$. If $r_{\mathcal{Q}}<1$, then $\|\mathbf{z}_i\|_c\le\|\bar{\mathbf{z}}_i\|_c\le\left\|\mathcal{Q}^i\right\|_B\|\mathbf{z}_0\|_c<Me^{-\omega i}\|\mathbf{z}_0\|_c$, where $\|\mathbf{z}_i\|_c=\|y_i\|_{[0,T]}+|x_i(0)|$. So, if $r_{\mathcal{Q}}<1$, then $\|y_i\|_{[0,T]}$, $|x_i(0)|$ converge to zero exponentially. □

3. Problem formulation

To illustrate the generality of the proposed RC scheme, we consider the following error dynamics examined in [3,4]:

$$\dot{\mathbf{e}}(t)=\mathbf{f}(t,\mathbf{e})+\mathbf{b}(t,\mathbf{e})\left(\mathbf{v}(t)-\hat{\mathbf{v}}(t)\right) \tag{6}$$

where $\mathbf{e}(t)\in\mathbb{R}^n$ is an error vector, $\mathbf{v}(t)=[v_1\ \cdots\ v_m]^{\mathrm{T}}\in\mathbb{R}^m$ is an unknown continuous-time vector function, $\hat{\mathbf{v}}(t)\in\mathbb{R}^m$ is a subsequently designed learning-based estimate of $\mathbf{v}(t)$ and $\mathbf{f}(t,\mathbf{e})\in\mathbb{R}^n$ and $\mathbf{b}(t,\mathbf{e})\in\mathbb{R}^{n\times m}$ are continuous vector functions with respect to the arguments t and $\mathbf{e}$. Moreover, we make the following assumption.

Assumption 1. The unknown continuous-time function vector $\mathbf{v}(t)$ is periodic, i.e.

$$\mathbf{v}(t+T)=\mathbf{v}(t)$$

where T is a known period. Moreover, $\mathbf{v}(t)$ is within the upper and lower bounds of a saturation function, namely

$$\mathbf{v}(t)=\mathrm{sat}(\mathbf{v}(t))$$

where

$$\mathrm{sat}(\boldsymbol{\xi})\triangleq[\mathrm{sat}_{\beta_1}(\xi_1)\ldots\mathrm{sat}_{\beta_m}(\xi_m)]^{\mathrm{T}}$$

$$\mathrm{sat}_{\beta_i}(\xi_i)=\begin{cases}\xi_i & \text{for } |\xi_i|\le\beta_i\\ \mathrm{sgn}(\xi_i)\,\beta_i & \text{for } |\xi_i|>\beta_i\end{cases},\ \forall\xi_i\in\mathbb{R}$$

and $\mathrm{sgn}(\cdot)$ denotes the standard signum function.

A lemma about the saturation is needed for the analysis in the next section, stated in the following.

Lemma 3 ([4]). *For any* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$, *the following inequality always holds*

$$\|sat(\mathbf{a}) - sat(\mathbf{b})\| \le \|\mathbf{a} - \mathbf{b}\|.$$

Although Lyapunov functions are certainly useful for this problem, they have particular difficulties at the same time. In most cases, although there exists a Lyapunov function suitable to a general system like (7), it is often difficult to find it. For example, under some conditions, if the trajectories of the system

$$\dot{\mathbf{e}}(t) = \mathbf{f}(t, \mathbf{e}) \tag{7}$$

satisfy (8)

$$\|\mathbf{e}(t)\| \le k\|\mathbf{e}(t_0)\| e^{-\lambda(t-t_0)}, \forall t \ge t_0 \ge 0 \tag{8}$$

where $k, \lambda \in \mathbb{R}_+$, then there exists a Lyapunov function by the converse Lyapunov theorem [18, pp. 163–165], which is formulated in an assumption in the following.

Assumption 2. The origin of the error system is globally exponentially stable for

$$\dot{\mathbf{e}}(t) = \mathbf{f}(t, \mathbf{e}). \tag{9}$$

Furthermore, there exist a first-order differentiable, positive -definite function $V(t,\mathbf{e})$ and positive constants $c_i, i = 1, \ldots, 4$, such that

$$c_1\|\mathbf{e}(t)\|^2 \le V(t,\mathbf{e}) \le c_2\|\mathbf{e}(t)\|^2 \tag{10}$$

$$\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \mathbf{e}}\right)^{\mathrm{T}} \mathbf{f}(t, \mathbf{e}) \le -c_3\|\mathbf{e}(t)\|^2 \tag{11}$$

$$\left\|\frac{\partial V}{\partial \mathbf{e}}\right\| \le c_4\|\mathbf{e}(t)\|. \tag{12}$$

Assumption 3. There exist $\varsigma_l, \varsigma_h \in \mathbb{R}_+$ such that

$$0 < \varsigma_l \le \|\mathbf{b}(t, \mathbf{e})\| \le \varsigma_h, \forall \mathbf{e} \in \mathbb{R}^n, \forall t \in \mathbb{R}_+.$$

Assumption 4. The function $\mathbf{f}(t, \mathbf{e})$ is a continuously differentiable function with respect to the arguments t and $\mathbf{e}$. Furthermore

$$\left\|\frac{\partial \mathbf{f}(t, \mathbf{e})}{\partial \mathbf{e}}\right\| \le l, \forall t \in \mathbb{R}_+. \tag{13}$$

Under Assumptions 1–4, the objective is to design a $\hat{\mathbf{v}}(t) \in \mathbb{R}^m$ subject to the saturation sat$(\cdot)$ for the system (6) to make $\mathbf{e}(t)$ tend to zero exponentially.

Remark 1. Assumption 2 is used to replace the inequality (8). In fact, the positive-definite function $V(t,\mathbf{e})$ does not need to be known in the RC design. If the origin of the error dynamics (9) is globally exponentially stable and the trajectories of the system satisfy (8), then by the converse Lyapunov theorem, there exists $V(t,\mathbf{e})$ such that $c_i, i = 1, \ldots, 4$ can be chosen as [18, pp. 163–165]

$$c_1(\delta) = \frac{1 - e^{-2l\delta}}{2l}, c_2(\delta) = \frac{k^2\left(1 - e^{-2\lambda\delta}\right)}{2\lambda}$$
$$c_3(\delta) = 1 - k^2 e^{-2\lambda\delta}, c_4(\delta) = \frac{2k}{\lambda - l}\left(1 - e^{-(\lambda-l)\delta}\right) \tag{14}$$

where δ is a positive constant to be chosen and l is defined in (13).

Remark 2. This remark is to show that Assumption 4 is reasonable. Under Assumption 1, $\|\mathbf{v}(t)\| \le b_\beta, \forall t \in \mathbb{R}_+$, where $b_\beta = \left\|\begin{bmatrix}\beta_1 & \beta_2 & \cdots & \beta_m\end{bmatrix}^{\mathrm{T}}\right\|$. Since the learning term $\hat{\mathbf{v}}(t) \in \mathbb{R}^m$ is designed subject to the saturation sat$(\cdot)$, the term satisfies

$$\left\|\mathbf{b}(t, \mathbf{e})\left(\mathbf{v}(t) - \hat{\mathbf{v}}(t)\right)\right\| \le 2\varsigma_h b_\beta \tag{15}$$

$\forall \mathbf{e} \in \mathbb{R}^n, \forall t \in \mathbb{R}_+$, where Assumption 3 is utilized. Based on (15) and Assumption 2, taking the derivative of V defined in Assumption 2 along (6) results in

$$\dot{V} \le -c_3\|\mathbf{e}(t)\|^2 + 2c_4\|\mathbf{e}(t)\| \varsigma_h b_\beta.$$

By [18, Theorem 4.18, p. 172], $\mathbf{e}(t)$ is uniformly bounded. According to this fact, l in (13) depends on the bound on $\mathbf{e}(t)$. In this case, the term $\mathbf{f}(t, \mathbf{e})$ is only required to satisfy the local Lipschitz condition rather than the global Lipschitz condition. Particularly, if the term $\mathbf{f}(t, \mathbf{e})$ satisfies the global Lipschitz condition, then l depends on the Lipschitz constant directly.

4. Saturated D-type RC design and convergence analysis

4.1. Saturated D-type RC design

The saturated D-type RC is proposed as follows:

$$\hat{\mathbf{v}}(t) = \mathrm{sat}\left(\hat{\mathbf{v}}(t - T) + \mathbf{h}(t - T)\right)$$
$$\hat{\mathbf{v}}(t) = \mathbf{0}, \forall t \in [-T, 0). \tag{16}$$

Here $\mathbf{h}(t) = \boldsymbol{\Gamma}(t)\left(\dot{\mathbf{e}}(t) - \bar{\mathbf{f}}(t, \mathbf{e})\right)$, where $\bar{\mathbf{f}}(t, \mathbf{e})$ is assumed to satisfy

$$\left\|\mathbf{f}(t, \mathbf{e}) - \bar{\mathbf{f}}(t, \mathbf{e})\right\| \le \gamma\|\mathbf{e}(t)\|, \gamma \in \mathbb{R}_+ \tag{17}$$

and $\boldsymbol{\Gamma}(t) \in \mathbb{R}^{m\times m}$, a continuous-time matrix on $[-T, \infty)$ with $\sup_t \|\boldsymbol{\Gamma}(t)\| \le k_{\Gamma} < \infty$, is to make $\hat{\mathbf{v}}(t)$ continuous on $[0, \infty)$.

Remark 3. In most cases, $\mathbf{f}(t, \mathbf{e})$ and $\mathbf{b}(t, \mathbf{e})$ can be written as $\mathbf{f}(t, \mathbf{e}) = \bar{\mathbf{f}}(t, \mathbf{e}) + \Delta\mathbf{f}(t, \mathbf{e})$ and $\mathbf{b}(t, \mathbf{e}) = \bar{\mathbf{b}}(t, \mathbf{e}) + \Delta\mathbf{b}(t, \mathbf{e})$, where $\bar{\mathbf{f}}, \bar{\mathbf{b}}$ and $\Delta\mathbf{f}, \Delta\mathbf{b}$ denote certain terms and uncertain terms, respectively. Meanwhile, $\bar{\mathbf{f}}(t, \mathbf{e})$ can also be considered as an approximate function vector of $\mathbf{f}(t, \mathbf{e})$ with γ measuring the approximate degree between them.

Remark 4. In the following design, the parameters in Assumptions 2–4 and k, λ in (8) will be taken as conditions for convergence analysis and will not appear in the controller. The given controller is simple with fewer parameters, which can be tuned directly in practice.

Remark 5. It is easier to estimate $\dot{\mathbf{e}}(t - T)$ than $\dot{\mathbf{e}}(t)$ at time t. The term $\dot{\mathbf{e}}(t)$ only can use $\mathbf{e}(t - iT_s)$ to estimate $\dot{\mathbf{e}}(t)$, where $T_s \in \mathbb{R}_+$ is the sampling period, $i = 0, 1, \ldots$. The Laplace transform of the derivative, namely s, is not a proper transfer function,[1] which cannot be realized physically. So, the approximation is often needed. However, the term $\dot{\mathbf{e}}(t - T)$ can be estimated by $\mathbf{e}(t - T - iT_s)$, $i = 0, \pm 1, \ldots, \pm\lfloor T/T_s\rfloor$. This can be realized physically. Adequate calculation can be applied to obtaining the derivative of $\dot{\mathbf{e}}(t - T)$ with good precision [19, p. 4], [20].

4.2. Convergence analysis

By using Assumptions 2–3 and the dynamics (6), the following inequality is obtained as

$$\dot{V}(t, \mathbf{e}) = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \mathbf{e}}\right)^{\mathrm{T}} \mathbf{f}(t, \mathbf{e}) + \left(\frac{\partial V}{\partial \mathbf{e}}\right)^{\mathrm{T}} \mathbf{b}(t, \mathbf{e})\tilde{\mathbf{v}}(t)$$

1 A proper transfer function is a transfer function in which the order of the numerator is not greater than the order of the denominator.

$$\le -c_3 \|\mathbf{e}(t)\|^2 + \left(\frac{\partial V}{\partial \mathbf{e}}\right)^{\mathrm{T}} \mathbf{b}(t,\mathbf{e})\,\tilde{\mathbf{v}}(t)$$
$$\le -\frac{c_3}{c_2} V(t,\mathbf{e}) + \frac{c_4}{\sqrt{c_1}} \varsigma_h \sqrt{V(t,\mathbf{e})}\,\|\tilde{\mathbf{v}}(t)\|.$$

where $\tilde{\mathbf{v}} \triangleq \mathbf{v} - \hat{\mathbf{v}}$. By defining $U(t,\mathbf{e}) \triangleq \sqrt{V(t,\mathbf{e})}$ and using the above inequality, it can be easily verified that [3], [18, p. 203]

$$\mathcal{D}^+ U(t,\mathbf{e}) \le -b_1 U(t,\mathbf{e}) + b_2 \|\tilde{\mathbf{v}}(t)\|$$

where $b_1 = \frac{c_3}{2c_2}$ and $b_2 = \frac{c_4}{2\sqrt{c_1}}\varsigma_h$. Furthermore, the following inequality results from the comparison lemma [18, pp. 102–103] as

$$U(t,\mathbf{e}) \le e^{-b_1(t-t_0)} U(t_0,\mathbf{e}) + b_2 \int_{t_0}^{t} e^{-b_1(t-s)} \|\tilde{\mathbf{v}}(s)\|\,\mathrm{d}s. \tag{18}$$

On the other hand, let $\boldsymbol{\phi}(t) = \hat{\mathbf{v}}(t-T) + \mathbf{h}(t-T)$. By Assumption 1 and the definition of $\mathbf{h}(t)$, one has

$$\begin{aligned}\mathbf{v}(t) - \boldsymbol{\phi}(t) &= \tilde{\mathbf{v}}(t-T) - \mathbf{h}(t-T)\\ &= (\mathbf{I}_m - \boldsymbol{\Gamma}(t-T)\mathbf{b}(t-T,\mathbf{e}))\tilde{\mathbf{v}}(t-T)\\ &\quad - \boldsymbol{\Gamma}(t-T)\left(\mathbf{f}(t-T,\mathbf{e}) - \bar{\mathbf{f}}(t-T,\mathbf{e})\right).\end{aligned} \tag{19}$$

Taking norm $\|\cdot\|$ on both sides of the above equation yields

$$\|\mathbf{v}(t) - \boldsymbol{\phi}(t)\| \le b_3 \|\tilde{\mathbf{v}}(t-T)\| + \gamma k_{\boldsymbol{\Gamma}} \|\mathbf{e}(t-T)\| \tag{20}$$

where $b_3 = \sup_t \|\mathbf{I}_m - \boldsymbol{\Gamma}(t)\mathbf{b}(t,\mathbf{e})\|$. Since $\mathbf{v}(t) = \mathrm{sat}(\mathbf{v}(t))$ by Assumption 1 and $\hat{\mathbf{v}}(t) = \mathrm{sat}(\boldsymbol{\phi}(t))$ according to (16), one has

$$\|\tilde{\mathbf{v}}(t)\| \le \|\mathbf{v}(t) - \boldsymbol{\phi}(t)\|$$

by Lemma 3. Then, by using the inequality (20), the inequality above is further bounded as

$$\|\tilde{\mathbf{v}}(t)\| \le b_3 \|\tilde{\mathbf{v}}(t-T)\| + \gamma k_{\boldsymbol{\Gamma}} \|\mathbf{e}(t-T)\| \tag{21}$$

by using (19). Using Assumption 2, the inequality (21) further becomes

$$\|\tilde{\mathbf{v}}(t)\| \le b_3 \|\tilde{\mathbf{v}}(t-T)\| + b_4 U(t-T,\mathbf{e}) \tag{22}$$

where $b_4 = \frac{\gamma k_{\boldsymbol{\Gamma}}}{\sqrt{c_1}}$. Define $\mathbf{e}_i(\tau) \triangleq \mathbf{e}(iT+\tau)$, $\tilde{\mathbf{v}}_i(\tau) \triangleq \tilde{\mathbf{v}}(iT+\tau)$, and $U_i(\tau,\mathbf{e}) \triangleq U(iT+\tau,\mathbf{e})$, where $i = 1, 2, \ldots$. Then, inequalities (18) and (22) become

$$\begin{aligned}&U_i(\tau,\mathbf{e}) \le e^{-b_1\tau} U_i(0,\mathbf{e}) + b_2 \int_0^{\tau} e^{-b_1(\tau-s)} \|\tilde{\mathbf{v}}_i(s)\|\,\mathrm{d}s\\ &\|\tilde{\mathbf{v}}_{i+1}(\tau)\| \le b_3 \|\tilde{\mathbf{v}}_i(\tau)\| + b_4 U_i(\tau,\mathbf{e}).\end{aligned} \tag{23}$$

Based on Lemma 2, the following theorem is obtained.

Theorem 1. *For the system (6) under Assumptions 1–4, suppose the saturated RC is designed as in (16), with the parameters satisfying*

$$\begin{aligned}&\sup_t \|\mathbf{I}_m - \boldsymbol{\Gamma}(t)\mathbf{b}(t,\mathbf{e})\| < 1\\ &\frac{\varsigma_h \gamma k_{\boldsymbol{\Gamma}}}{1 - \sup_t \|\mathbf{I}_m - \boldsymbol{\Gamma}(t)\mathbf{b}(t,\mathbf{e})\|} < \frac{c_1 c_3}{c_2 c_4}.\end{aligned} \tag{24}$$

Then $\mathbf{e}(t) = \mathbf{0}$ of (6) is globally exponentially stable.

Proof. Based on (23), $\mathbf{e}(t) = \mathbf{0}$ of (6) is globally exponentially stable if

$$b_3 < 1 \text{ and } \frac{b_2 b_4}{b_1(1-b_3)} < 1.$$

One further has

$$b_3 < 1 \Leftrightarrow \sup_t \|\mathbf{I}_m - \boldsymbol{\Gamma}(t)\mathbf{b}(t,\mathbf{e})\| < 1$$
$$\frac{b_2 b_4}{b_1(1-b_3)} < 1 \Leftrightarrow (24). \quad \square$$

The next step is to replace $\frac{c_1 c_3}{c_2 c_4}$ in (24) by k, λ, l, δ. Before proceeding further with the development of this work, the following lemma is needed.

Lemma 4. *Under Assumptions 2, 4, if the trajectories of the system (9) satisfy (8), then $0 < \lambda \le l$.*

Proof. This proof can be shown by contradiction. By using Assumption 4, $\mathbf{e}(t)$, the solution of (9), satisfies that $e^{-l(t-t_0)}\|\mathbf{e}(t_0)\| \le \|\mathbf{e}(t)\| \le \|\mathbf{e}(t_0)\| e^{l(t-t_0)}$ [18, p. 107]. Since the trajectories of the system (9) satisfy (8), one has $e^{(\lambda-l)(t-t_0)} \le k$. If $l < \lambda$, then $k < e^{(\lambda-l)(t-t_0)}$ when $t \ge t_0 + \frac{1}{\lambda-l}\ln k$. This contradicts $e^{(\lambda-l)(t-t_0)} \le k$. Thus $\lambda \le l$. $\square$

Theorem 2. *The origin of the error system (9) is globally exponentially stable with the trajectories satisfying (8). Considering the system (6) with Assumptions 1–4 under the control law (16), if*

$$\begin{aligned}&\sup_t \|\mathbf{I}_m - \boldsymbol{\Gamma}(t)\mathbf{b}(t,\mathbf{e})\| < 1\\ &\frac{\varsigma_h \gamma k_{\boldsymbol{\Gamma}}}{1 - \sup_t \|\mathbf{I}_m - \boldsymbol{\Gamma}(t)\mathbf{b}(t,\mathbf{e})\|} < g(\delta^*)\end{aligned} \tag{25}$$

then $\mathbf{e}(t) = \mathbf{0}$ of (6) is globally exponentially stable. Here $\delta^* = \arg\max_{\delta > \frac{\ln k}{\lambda}} g(\delta)$, where

$$g(\delta) = \frac{c_1(\delta)\,c_3(\delta)}{c_2(\delta)\,c_4(\delta)} \tag{26}$$

with $c_1, \ldots, c_4$ defined in (14).

Proof. The free parameter δ in (6) can be chosen through the solution to an optimization problem as follows:

$$\begin{aligned}\max_{\delta}\, &g(\delta) = \frac{c_1(\delta)\,c_3(\delta)}{c_2(\delta)\,c_4(\delta)}\\ \text{s.t.}\quad &c_2(\delta) - c_1(\delta) \ge 0\\ &c_i(\delta) > 0 \quad i = 1,\ldots,4\\ &\delta > 0\end{aligned}. \tag{27}$$

Since $k \ge 1$ (note that (8) when $t = t_0$) and $0 < \lambda \le l$ by Lemma 4, the inequality $c_1(\delta) \le c_2(\delta)$ and the inequalities $c_i(\delta) > 0, i = 1, 2, 3, 4$ will always be satisfied with $\delta > 0$. If δ satisfies $\delta > \frac{\ln k}{\lambda}$, then $c_3(\delta) > 0$. Therefore, the optimization problem (27) is simplified as follows:

$$\max_{\delta > \frac{\ln k}{\lambda}} g(\delta) = \frac{c_1(\delta)\,c_3(\delta)}{c_2(\delta)\,c_4(\delta)}.$$

Let $\delta^* = \arg\max_{\delta > \frac{\ln k}{\lambda}} g(\delta)$. If (25) holds, then there exist $c_i(\delta^*)$, $i = 1, 2, 3, 4$ satisfying (24). The proof is now concluded by an application of Theorem 1. $\square$

Remark 6. In light of the idea in [15], the saturated D-type RC control law (16) can also be updated based only on the steady-state output so that the convergence condition is more relaxed and monotonic convergence in the repetitive domain can be achieved. In this case, only the condition $\sup_t \|\mathbf{I}_m - \boldsymbol{\Gamma}(t)\mathbf{b}(t,\mathbf{e})\| < 1$ needs to be retained. Interested readers can refer to [15] for details. However, the price to be paid is the increase in number of repetitive trials.

5. Robotic manipulator tracking example

5.1. Problem formulation

The dynamics of an m-degree-of-freedom manipulator are described by the following differential equation

$$\mathbf{D}(\mathbf{q}(t))\,\ddot{\mathbf{q}}(t) + \mathbf{C}(\mathbf{q}(t),\dot{\mathbf{q}}(t))\,\dot{\mathbf{q}}(t) + \mathbf{G} = \boldsymbol{\tau}(t) + \mathbf{v}(t) \tag{28}$$

where $\mathbf{q}(t) \in \mathbb{R}^m$ denotes the vector of generalized displacements in robot co-ordinates, $\boldsymbol{\tau}(t) \in \mathbb{R}^m$ denotes the vector of generalized control input forces in robot coordinates; $\mathbf{D}(\mathbf{q}(t)) \in \mathbb{R}^{m\times m}$ is the manipulator inertial matrix, $\mathbf{C}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) \in \mathbb{R}^{m\times m}$ is the vector of centripetal and Coriolis torques and $\mathbf{G}(\mathbf{q}(t)) \in \mathbb{R}^m$ is the vector of gravitational torques; $\mathbf{v} \in \mathcal{C}^1_{PT}([0,\infty); \mathbb{R}^m)$ is a T-periodic disturbance. It is assumed that only $\mathbf{q}(t)$ and $\dot{\mathbf{q}}(t)$ are available from measurements. Define a filtered tracking error as

$$\mathbf{e}(t) = \dot{\tilde{\mathbf{q}}}(t) + \tilde{\mathbf{q}}(t) \tag{29}$$

where $\tilde{\mathbf{q}}(t) = \mathbf{q}_{\rm d}(t) - \mathbf{q}(t)$, and $\mathbf{q}_{\rm d}(t)$ is a desired trajectory.

The following assumptions are needed, which are common to robot manipulators [3–5].

(A1) The inertial matrix $\mathbf{D}(\mathbf{q}(t))$ is symmetric, uniformly positive definite and bounded, i.e.,

$$0 < \underline{\lambda}_D \mathbf{I}_m \le \mathbf{D}(\mathbf{q}(t)) \le \bar{\lambda}_D \mathbf{I}_m < \infty, \forall \mathbf{q}(t) \in \mathbb{R}^m \tag{30}$$

where $\underline{\lambda}_D$ and $\bar{\lambda}_D$ are positive real numbers.

(A2) The matrix $\dot{\mathbf{D}}(\mathbf{q}(t)) - 2\mathbf{C}(\mathbf{q}(t), \dot{\mathbf{q}}(t))$ is skew-symmetric, hence

$$\mathbf{x}^{\rm T}\left(\dot{\mathbf{D}}(\mathbf{q}(t)) - 2\mathbf{C}(\mathbf{q}(t), \dot{\mathbf{q}}(t))\right)\mathbf{x} = 0, \forall \mathbf{x} \in \mathbb{R}^m.$$

For a given desired trajectory $\mathbf{q}_{\rm d} \in \mathcal{C}^2_{PT}([0,\infty); \mathbb{R}^m)$, our objective is to design a controller such that $\lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}$.

Remark 7. From (29), it is known that both $\tilde{\mathbf{q}}(t)$ and $\dot{\tilde{\mathbf{q}}}(t)$ can be viewed as outputs of a stable system with $\mathbf{e}(t)$ as input, which means that $\tilde{\mathbf{q}}(t)$ and $\dot{\tilde{\mathbf{q}}}(t)$ are bounded or approach zero if $\mathbf{e}(t)$ is bounded or approaches zero. Assumptions (A1)–(A2) are commonly satisfied by a robot manipulator.

5.2. Model transformation

Design $\boldsymbol{\tau}(t)$ as

$$\boldsymbol{\tau}(t) = \mathbf{D}(\mathbf{q}(t))\ddot{\mathbf{q}}_{\rm e}(t) + \mathbf{C}(\mathbf{q}(t), \dot{\mathbf{q}}(t))\dot{\mathbf{q}}_{\rm e}(t) + \mathbf{G} + \mathbf{P}\mathbf{e}(t) + \hat{\mathbf{v}}(t) \tag{31}$$

where $\dot{\mathbf{q}}_{\rm e}(t) = \dot{\mathbf{q}}_{\rm d}(t) + \tilde{\mathbf{q}}(t)$, $\ddot{\mathbf{q}}_{\rm e}(t) = \ddot{\mathbf{q}}_{\rm d}(t) + \dot{\tilde{\mathbf{q}}}(t)$, $\mathbf{P} \in \mathbb{R}^{m\times m}$ is a positive definite matrix, and $\hat{\mathbf{v}}(t) \in \mathbb{R}^m$ is the estimate of $\mathbf{v}(t)$. By employing (31), the filtered error dynamics are obtained as follows:

$$\mathbf{D}(\mathbf{q}(t))\dot{\mathbf{e}}(t) + \mathbf{C}(\mathbf{q}(t), \dot{\mathbf{q}}(t))\mathbf{e}(t) = -\mathbf{P}\mathbf{e}(t) + \left(\mathbf{v}(t) - \hat{\mathbf{v}}(t)\right). \tag{32}$$

Furthermore, the system above can be written in the form of (6) with

$$\begin{aligned}\mathbf{f}(t,\mathbf{e}) &= -\mathbf{D}^{-1}(\mathbf{q}(t))\left(\mathbf{C}(\mathbf{q}(t), \dot{\mathbf{q}}(t))\mathbf{e}(t) + \mathbf{P}\mathbf{e}(t)\right)\\ \mathbf{b}(t,\mathbf{e}) &= \mathbf{D}^{-1}(\mathbf{q}(t)).\end{aligned} \tag{33}$$

5.3. Verification of assumptions

Since $\mathbf{v} \in \mathcal{C}^1_{PT}([0,\infty); \mathbb{R}^m)$ is a T-periodic disturbance, Assumption 1 holds. Define a Lyapunov function

$$V(t,\mathbf{e}) = \frac{1}{2}\mathbf{e}^{\rm T}(t)\mathbf{D}(\mathbf{q}(t))\mathbf{e}(t). \tag{34}$$

Then

$$\frac{1}{2}\underline{\lambda}_D\|\mathbf{e}(t)\|^2 \le V(t,\mathbf{e}) \le \frac{1}{2}\bar{\lambda}_D\|\mathbf{e}(t)\|^2.$$

By skew-symmetry of matrix $\dot{\mathbf{D}}(\mathbf{q}(t)) - 2\mathbf{C}(\mathbf{q}(t), \dot{\mathbf{q}}(t))$, the time derivative of $V(t,\mathbf{e})$ along (6) with (33) is evaluated as

$$\dot{V}(t,\mathbf{e}) \le -\mathbf{e}^{\rm T}(t)\mathbf{P}\mathbf{e}(t) \tag{35}$$

$$\left\|\frac{\partial V}{\partial \mathbf{e}}\right\| \le \bar{\lambda}_D\|\mathbf{e}(t)\|. \tag{36}$$

Therefore, Assumption 2 is satisfied. Since

$$\frac{1}{\bar{\lambda}_D} \le \|\mathbf{b}(t,\mathbf{e})\| \le \frac{1}{\underline{\lambda}_D}$$

Assumption 3 holds. If $\mathbf{e}(t)$ is bound, then $\mathbf{q}(t), \dot{\mathbf{q}}(t)$ are bound by (29). In this case, Assumption 4 holds.

5.4. Control design

Design $\boldsymbol{\tau}(t)$ as

$$\begin{aligned}\boldsymbol{\tau}(t) &= \bar{\mathbf{D}}(\mathbf{q}(t))\ddot{\mathbf{q}}_{\rm e}(t) + \bar{\mathbf{C}}(\mathbf{q}(t), \dot{\mathbf{q}}(t))\dot{\mathbf{q}}_{\rm e}(t) + \bar{\mathbf{G}} + \mathbf{P}\mathbf{e}(t) + \hat{\mathbf{v}}(t)\\ \hat{\mathbf{v}}(t) &= \text{sat}\left(\hat{\mathbf{v}}(t-T) + \mathbf{h}(t-T)\right), \hat{\mathbf{v}}(t) = \mathbf{0}, \forall t \in [-T,0)\end{aligned} \tag{37}$$

where $\mathbf{h}(t) = \boldsymbol{\Gamma}(t)\left(\dot{\mathbf{e}}(t) + \bar{\mathbf{D}}^{-1}(\mathbf{q}(t))\left(\bar{\mathbf{C}}(\mathbf{q}(t), \dot{\mathbf{q}}(t))\mathbf{e}(t) + \mathbf{P}\mathbf{e}(t)\right)\right)$. Here $\bar{\mathbf{D}}, \bar{\mathbf{C}}, \bar{\mathbf{G}}$ are the plant parameters known to the designer.

5.5. Numerical simulation

The robot, the initial condition, tracking task and disturbance used for a 3-degree-of-freedom manipulator are the same as in [9], where parameters $J_p = 0.8$, $l_2 = 2$, $l_3 = 1$, and $g = 9.8$. However, they are only known approximately by the designer as follows:

$$\begin{aligned}&\bar{l}_2 = l_2 + 0.2 * \text{randn}, \bar{l}_3 = 1 + 0.2 * \text{randn},\\ &\bar{J}_p = J_p + 0.2 * \text{randn}, \bar{g} = 9.8\end{aligned} \tag{38}$$

where randn is a normally distributed random number with mean 0 and standard deviation 1 for simulation purpose. With them, $\bar{\mathbf{D}}, \bar{\mathbf{C}}, \bar{\mathbf{G}}$ are obtained. In (37), choose $\mathbf{P}$ to be $\mathbf{P} \equiv 10\mathbf{I}_3$ and design $\hat{\mathbf{v}}(t)$ according to (16) as

$$\hat{\mathbf{v}}(t) = \text{sat}\left(\hat{\mathbf{v}}(t-T) + \mathbf{h}(t-T)\right)$$

where

$$\begin{aligned}\mathbf{h}(t) = {}& 0.3k_{\rm e}(t)\bar{\mathbf{D}}^{-1}(\mathbf{q}(t))\left(\dot{\mathbf{e}}(t) + \bar{\mathbf{D}}^{-1}(\mathbf{q}(t))\right.\\ &\left.\left(\bar{\mathbf{C}}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) + \mathbf{P}\mathbf{e}(t)\right)\right)\end{aligned}$$

$$k_{\rm e}(t) = \begin{cases} 0 & t \in [-T,0) \\ \dfrac{2\pi t}{3} & t \in [0,T) \\ 1 & t \in [T,\infty), \end{cases} \quad \beta_1 = \beta_2 = \beta_3 = 10.$$

For tracking performance comparison, the performance index is introduced as $J_k \triangleq \sup_{t\in[(k-1)T,kT]}\left\|\tilde{\mathbf{q}}(t)\right\|$, where $k = 1, 2, \ldots$. As seen in Fig. 1, the performance index J_k approaches 0 as k increases with four different sets of parameters $(\bar{l}_2, \bar{l}_3, \bar{J}_p)$. These imply that $\tilde{\mathbf{q}}(t)$ approaches $\mathbf{0}$ as $t \to \infty$. This result is consistent with the conclusion in Theorem 1.

5.6. Discussion

For the considered problem or some linear systems with weak nonlinear terms, the (high gain) feedback control with nonlinear compensation can often make the error dynamics (9) exponentially stable (based on Lyapunov analysis). Readers also can refer to [21] for a similar example. Although Assumptions 1–4, in Section 5.3 are verified one by one based on the Lyapunov function (only used for analysis), it should be noticed that the information about the Lyapunov function does not appear in the designed controller (37). Honestly, the Lyapunov analysis cannot be replaced to prove the exponential stability in theory. But, in practice, the data-based method can be used to find more real parameters related to the exponential convergence. What is more, parameters in the proposed

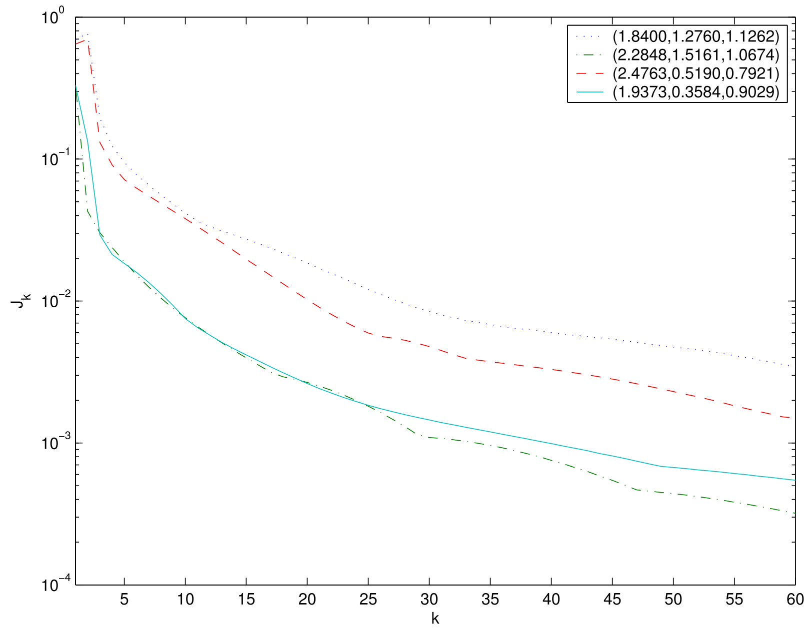

Fig. 1. Change of maximum Euclidean norm of error with the ith period with four different sets of parameters $(\bar{l}_2, \bar{l}_3, \bar{J}_p)$.

controller can be adjusted online, where $\bar{\mathbf{f}}(t, \mathbf{e})$ can be identified more and more accurate by using data.

In the following, the conditions in Theorem 2 will be verified only by the data acquired from the system. The inequality (8) can be written as

$$\ln \frac{\|\mathbf{e}(t)\|}{\|\mathbf{e}(0)\|} \leq \kappa - \lambda t, t \geq 0 \tag{39}$$

where $t_0 = 0$ and $\kappa = \ln k$. For the dynamics of the m-degree-of-freedom manipulator here, 50 trajectories of the corresponding error dynamics (9) are simulated and plotted in Fig. 2 with random initial conditions. As shown, the term $\ln(\|\mathbf{e}(t)\| / \|\mathbf{e}(0)\|)$ finally goes down linearly with respect to time t. This roughly implies that the origin of the error dynamics (9) is exponentially stable for this example. According to Fig. 2, the parameters are chosen as $k = 1.05$ and $\lambda = 3$. By simulation, one further has $\varsigma_h = 1$, $l = 10.1$ and $\sup_t \|\mathbf{I}_3 - 0.3\mathbf{b}(t, \mathbf{e})\| = 0.9$. With the parameters above, the function $g(\delta)$ defined in (26) is plotted in Fig. 3. As shown, $g(\delta)$ reaches the maximum $g(\delta^*) = 0.966$ with $\delta^* = 0.0463$. The condition (25) is satisfied when $\gamma < 0.322$. This gives the requirement on the nominal system $\bar{\mathbf{f}}(t, \mathbf{e})$.

6. Conclusion

This paper presents a saturated D-Type RC for a class of nonlinear systems. To bypass the difficulties in finding Lyapunov functions, a contraction mapping method based on spectral theory is proposed to analyze the closed-loop system. The proposed convergence condition for the closed-loop system depends on parameters of trajectories (refer to the definition of c_1, c_2, c_3, c_4 defined in (14)) rather than the concrete forms of the Lyapunov functions. The feasibility of our work is demonstrated through a robotic manipulator tracking example. Since the analysis for the nonlinear RC system is similar to that of ILC scheme, it bridge the gap between RC and ILC.

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Appendix

(1) Condition conversion: if and only if (5) *holds, then* $\Phi(z) \neq 0$, $\forall |z| \geq 1, z \in \mathbb{C}$, *where* $\Phi(z)$ *is defined in* (48). From the compact maps of spectral theory [17, p. 238, section 21.2], [22], it can be concluded that either $r_{\mathcal{Q}} < 1$ or $\exists z_1 \in \sigma(\mathcal{Q}), |z_1| \geq 1$ is satisfied. If the case of $|z| \geq 1$ is excluded, then $r_{\mathcal{Q}} < 1$ holds. It implies $r_{\mathcal{Q}} < 1$ if and only if the solution $\mathbf{w}$ to the following equation

$$(\mathcal{Q} - z\mathbf{I})\mathbf{w} = \mathbf{0}, \forall |z| \geq 1 \tag{40}$$

is unique zero, where

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{R} \times \mathcal{C}([0, T], \mathbb{R})$$

and $\mathbf{I}$ and $\mathbf{0}$ are unit operator and zero respectively in $\mathbb{R} \times \mathcal{C}([0, T], \mathbb{R})$. Before proceeding further, Eq. (40) is converted into an equivalent form that is more convenient for the proofs of sufficiency and necessity. Writing out (40) yields

$$e^{-a_1 T} w_1 - z w_1 + a_2 \int_0^T e^{-a_1(T-s)} w_2(s)\, ds = 0 \tag{41}$$

$$a_4 e^{-a_1 \tau} w_1 + a_3 w_2(\tau) - z w_2(\tau) + a_2 a_4 \int_0^\tau e^{-a_1(\tau-s)} w_2(s)\, ds = 0 \tag{42}$$

where $\tau \in [0, T]$. Let

$$\mu(\tau) = e^{-a_1 \tau} w_1 + a_2 \int_0^\tau e^{-a_1(\tau-s)} w_2(s)\, ds, \tau \in [0, T].$$

Then

$$\dot{\mu}(\tau) = -a_1 \mu(\tau) + a_2 w_2(\tau). \tag{43}$$

By noticing the definition of $\mu(\tau)$ and (41), it is easy to get

$$\mu(0) = w_1, \mu(T) = \lambda w_1. \tag{44}$$

On the other hand, Eq. (42) becomes

$$w_2(\tau) = \frac{a_4}{z - a_3} \mu(\tau). \tag{45}$$

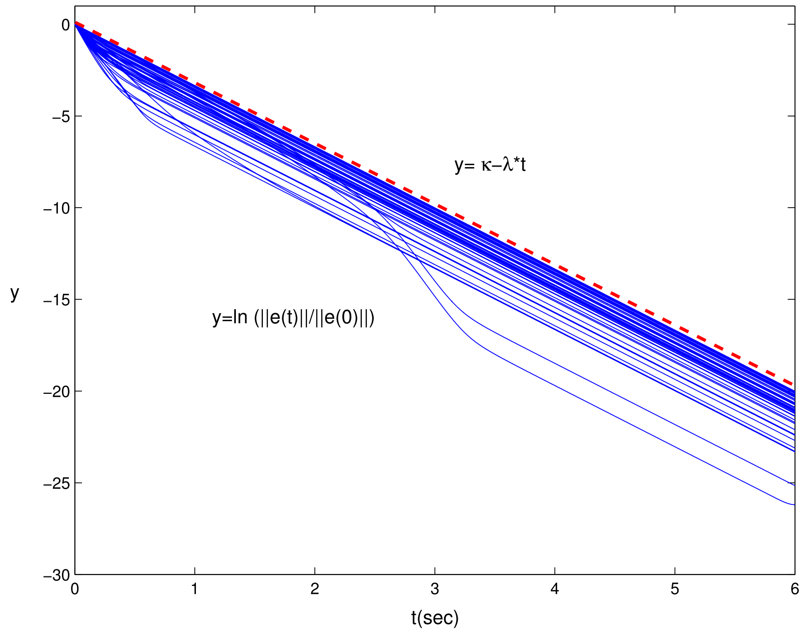

Fig. 2. The term $\ln \left\|\mathbf{e}\left(t\right)\right\| / \left\|\mathbf{e}\left(0\right)\right\|$ finally goes down linearly with respect to time t.

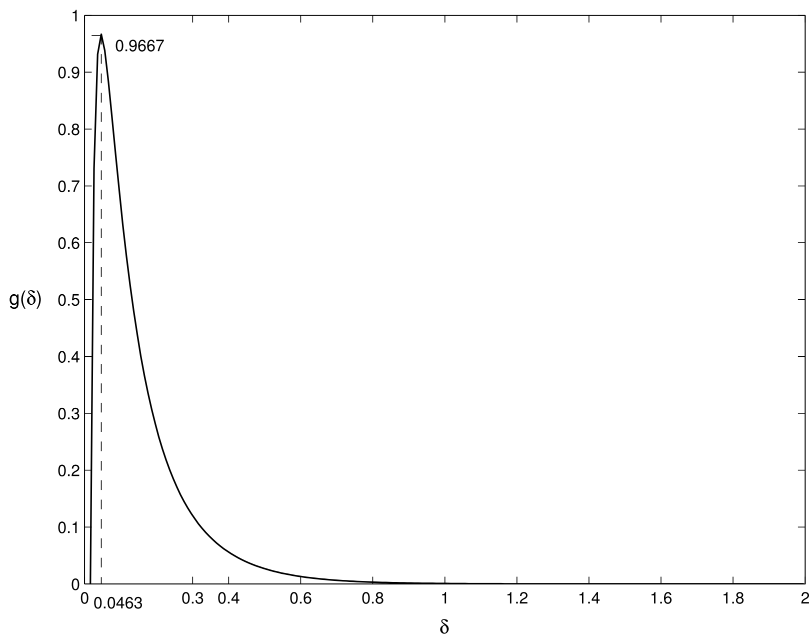

Fig. 3. The function $g\left(\delta\right)$ varies with respect to δ.

Combining (43) and (45) results in

$$\dot{\mu}\left(\tau\right) = -\left(a_1 - a_2 \frac{a_4}{z - a_3}\right)\mu\left(\tau\right).$$

Consequently,

$$\mu\left(\tau\right) = e^{-\left(a_1 - a_2 \frac{a_4}{z - a_3}\right)\tau}\mu\left(0\right). \tag{46}$$

Finally, by using (44), Eq. (46) further becomes

$$\Phi\left(z\right) w_1 = 0 \tag{47}$$

where

$$\Phi\left(z\right) \triangleq 1 - \frac{1}{z} e^{-\left(a_1 - a_2 \frac{a_4}{z - a_3}\right)T}. \tag{48}$$

And, using (44), (45), (46) yields

$$w_2\left(\tau\right) = \frac{a_4}{z - a_3} e^{-\left(a_1 - a_2 \frac{a_4}{z - a_3}\right)\tau} w_1. \tag{49}$$

Observed from (47), if $\Phi\left(z\right) \neq 0, \forall \left|z\right| \geq 1$, then the solution w_1 to Eq. (47) is zero and $w_2\left(\tau\right) \equiv 0$ consequently according to (49). This implies $r_{\mathcal{Q}} < 1$ if and only if $\Phi\left(z\right) \neq 0, \forall \left|z\right| \geq 1$. Therefore, the lemma is rephrased equivalently that *if and only if* (5) *holds, then* $\Phi\left(z\right) \neq 0, \forall \left|z\right| \geq 1$.

(2) Proof of sufficiency: if (5) *holds, then* $\Phi\left(z\right) \neq 0, \forall \left|z\right| \geq 1$. If $a_3 < 1$, then the definition (48) makes sense as $z - a_3 \neq 0$, $\forall \left|z\right| \geq 1$. Let $z = \alpha + \beta i$, where $\alpha, \beta \in \mathbb{R}$. For $\forall T > 0$, $\left|z\right| \geq 1$, one has

$$\begin{aligned}\left|\Phi\left(z\right)\right| &\geq 1 - \left|\frac{1}{z} e^{-\left(a_1 - a_2 \frac{a_4}{z - a_3}\right)T}\right| \\ &\geq 1 - e^{-q\left(\alpha,\beta\right)T}\end{aligned} \tag{50}$$

where $q\left(\alpha, \beta\right) = a_1 - a_2 a_4 \frac{\alpha - a_3}{\left(\alpha - a_3\right)^2 + \beta^2}$. It is easy to verify that $1 - e^{-q\left(\alpha,\beta\right)T}$ achieves the minimum at $\alpha^* = 1, \beta^* = 0$. Thus the inequality $1 - e^{-q\left(\alpha^*,\beta^*\right)T} > 0$ implies $\Phi\left(z\right) \neq 0$ by (50).

Simplifying $1-e^{-q(\alpha^*,\beta^*)T}>0$ results in

$$\frac{a_2a_4}{a_1\left(1-a_3\right)}<1. \tag{51}$$

This concludes the proof of sufficiency.

(3) Proof of necessity: if $\Phi(z)\neq 0, \forall |z|\geq 1$, then (5) *holds*. This necessary condition is proved by contradiction, i.e., *there exists a z^* such that $|z^*|\geq 1$ and $\Phi(z^*)=0$, if $a_3\geq 1$ or $\frac{a_2a_4}{a_1(1-a_3)}\geq 1$*. The condition ($a_3\geq 1$ or $\frac{a_2a_4}{a_1(1-a_3)}\geq 1$) is covered by two cases, namely *Case 1:* $a_3\geq 1$ and *Case 2:* $a_3<1$ and $\frac{a_2a_4}{a_1(1-a_3)}\geq 1$. As a result, the proof of necessity can be divided into the proof of Case 1 and the proof of Case 2. *(1) Proof of Case 1:* $a_3\geq 1$. Let $z_1=a_3+\epsilon_{a_3}$, $z_2=\frac{1}{\epsilon_{a_3}}$, $z_1,z_2\in\mathbb{R}_+$. For any $T>0$, if $\epsilon_{a_3}>0$ is small enough, then

$$\Phi\left(z_1\right)=1-\frac{1}{a_3+\epsilon_{a_3}}e^{-\left(a_1-a_2\frac{a_4}{\epsilon_{a_3}}\right)T}<0$$

$$\Phi\left(z_2\right)=1-\epsilon_{a_3}e^{-\left(a_1-a_2\frac{a_4}{\frac{1}{\epsilon_{a_3}}-a_3}\right)T}>0.$$

Since $\Phi(z)$ is continuous with respect to $z\in\mathbb{R}$ and $1<z_1<z_2$, there always exists a $z^*\in(z_1,z_2)$ such that $\Phi(z^*)=0$ according to the *mean value theorem*. *(2) Proof of Case 2:* $\frac{a_2a_4}{a_1(1-a_3)}\geq 1$ and $a_3<1$. In this case, $\Phi(1)=1-e^{-\left(a_1-a_2\frac{a_4}{1-a_3}\right)T}\leq 0$. On the other hand, $\Phi(z_0)=1-\frac{1}{z_0}e^{-\left(a_1-a_2\frac{a_4}{z_0-a_3}\right)T}>0$ with a large enough $z_0\in\mathbb{R}_+$. Therefore, there always exists a $z^*\in[1,z_0)$ such that $\Phi(z^*)=0$. Proofs of *Cases 1–2* conclude the proof of necessity.

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