

Engineering Notes

Optimization of Multicopter Propulsion System Based on Degree of Controllability

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Nomenclature

$B_{p,i}$	=	number of blades of propeller i
$c_{M,i}$	=	torque coefficient of propulsor i ; $c_{M,i} \in \mathbb{R}_+$
$c_{T,i}$	=	thrust coefficient of propulsor i ; $c_{T,i} \in \mathbb{R}_+$
$D_{p,i}$	=	diameter of propeller i , m
$d_{m,i}$	=	diameter of motor i , m
g	=	acceleration of gravity, $(\text{kg} \cdot \text{m})/\text{s}^2$
$H_{p,i}$	=	geometric pitch of propeller i , m
$I_{m0,i}$	=	no-load current of motor i , A
$i_{m,i}$	=	current of motor i , A
J_i	=	total inertia of motor i , $\text{kg} \cdot \text{m}^2$
J_{xx}, J_{yy}, J_{zz}	=	moments of inertia around the roll, pitch, and yaw axes of the multicopter frame, $\text{kg} \cdot \text{m}^2$
$K_{e,i}$	=	back-electromotive force constant of motor i
$K_{t,i}$	=	torque constant of motor i
$K_{V0,i}$	=	KV value of motor i , rpm/V
$k_{\mu,i}$	=	ratio between the reactive torque and thrust of propulsor i
M_i	=	antitorque of each propulsor i , $\text{N} \cdot \text{m}$
$M_{m,i}$	=	torque of motor i , $\text{N} \cdot \text{m}$
m_a	=	mass of the multicopter, kg
$m_{m,i}$	=	mass of motor i , kg
n_d	=	number of propulsors to be designed
$\mathbf{n}_i$	=	parameter vector of the i th propulsor
n_p	=	propulsor numbers of a multicopter
n_r	=	number of propulsor physical parameters
p_z	=	altitude of the multicopter, m
$R_{m,i}$	=	armature resistance of motor i , Ω
T_i	=	thrust of propulsor i , N
$T_{ss,i}$	=	thrust of propulsor i when the vehicle is in a hovering state, N
t_r	=	recovery time from current state to zero for a linear system
U_b	=	battery output voltage, V
$U_{m,i}$	=	voltage of motor i , V
$U_{m0,i}$	=	no-load motor input voltage of motor i , V
v_z	=	vertical velocity of the multicopter, m/s
ρ_{doc}	=	degree of controllability
σ_i	=	throttle command

ϕ, θ, ψ	=	roll, pitch, and yaw angles of the multicopter, rad
$\omega_x, \omega_y, \omega_z$	=	roll, pitch, and yaw angular velocities of the multicopter, rad/s
ω_i	=	angular speed of motor i , rad/s

I. Introduction

MULTICOPTERS have been attracting increasing attention in recent years due to their critical role in military and civil applications [1–3]. However, the selection of a propulsion system that will provide the desired performance is still one of the most daunting tasks in the process of developing a multicopter [4]. To design a multicopter, a designer may have to select the propulsion system [including the battery packs, electronic speed controls (ESCs), motors, and propellers] from a set of data sheets to assemble the multicopter. Then, simulations and experiments are conducted to determine whether the multicopter has met the performance requirements, e.g., endurance time and payload capacity. In practice, the design of a multicopter generally involves an iterative process of design and verification; namely, the issues of design and verification are intertwined. This situation makes the selection of the propulsion system difficult and time consuming, especially for inexperienced designers. Therefore, devoting efforts to simplify the design process of multicopters is a worthwhile endeavor.

There are a lot of ways to design multicopters, and a few concepts for the design optimization of multicopters based on a computer-aided solver have been presented to simplify the design process of multicopters. Based on a detailed model of the propulsion system, one can evaluate the performance (endurance time and payload capacity) of the considered multicopters [4,5]. Then, a propulsor optimizer can be used to design and optimize the propulsion system of the multicopters [4]. The optimal propulsion system will be obtained, depending on the objective (e.g., lightest weight, longest flight time, highest efficiency, or largest payload capacity) of the optimizer. In Ref. [6], six criteria were provided as possible objective function candidates when designing a multicopter, i.e., the minimum time constant, power consumption, price, inertia roll, and inertia pitch; or the maximum flight time. In practice, the designer needs to choose proper objective functions based on the application's specifications. Current propulsion system design methods mainly focus on achieving the mission requirements (e.g., payload capacity [7], power efficiency or long duration [8,9], and dynamic performance [10]) for which the vehicle is being designed. Multidisciplinary optimization approaches were also considered in Ref. [11]. Unquestionably, it has long been known that control performance is an important property of a plant. However, few works consider the control performance when selecting the appropriate propulsion systems for multicopters. If the considered multicopter is not sufficiently controllable, then the propulsion system needs to be redesigned according to specific control requirements, which are a waste of time and money. This Note will select the propulsion system for a multicopter by taking the control performance into consideration.

In this Note, a degree of controllability (DOC)-based methodology for optimizing the propulsion system of a multicopter is presented. The output of the methodology is generally a propulsion system that will allow a multicopter to have more control capacity (more capable of changing the system state). This type of methodology was first used to solve the problem of choosing actuator locations for the attitude and shape control of large flexible space structures [12–14]. Many DOCs were defined during the 1970s ~ 1980s: e.g., DOCs based on the system Gramian matrix [15] (called the Gramian-matrix-based DOC here), and DOCs based on the size of the region in

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the state space that can be returned to the origin in a prescribed time using bounded controls (called the state-norm-based DOC here) [16]. The readers are referred to a recent paper [17] for an outline of the state of the art in the field of the DOC. Compared with the Gramian-matrix-based DOC, the state-norm-based DOC can address control constraints. Therefore, this Note uses the state-norm-based DOC to optimize the propulsion system of a multicopter, being the first trial in the literature to the authors' best knowledge and aiming to obtain an optimal propulsion system that makes the considered multicopter more controllable.

The main contributions of this Note are as follows. The theoretical contributions are as follows:

1) The multicopter dynamics are presented, where the physical parameters of motors/propellers are undetermined variables.

2) A DOC-based methodology for multicopter design is proposed and verified, based on which one can design a multicopter with more control capacity.

The practical contributions are as follows:

1) A step-by-step design process is presented for multicopter design based on the DOC.

2) The optimal designs of a coaxial hexacopter with different vehicle masses are given, based on the proposed design process.

In Sec. II, the abstract model of a multicopter is presented, and some preliminaries on the state-norm-based DOC are provided. Then, multicopter modeling with propulsor dynamics is derived, and the computation method of the DOC for the considered multicopter is provided in Sec. III. In Sec. IV, an optimized design process for multicopter design based on the DOC is presented. Then, the step-by-step process is used to design a coaxial hexacopter with different desired vehicle masses to demonstrate its effectiveness. Finally, conclusions are drawn in Sec. V.

II. Problem Formulation

According to Ref. [12], the DOC-based methodology requires an appropriate criterion to represent the desirability of the propulsion system, a simple method to evaluate this criterion, and an algorithm to optimize the criterion over the space of possible propulsor parameters. Based on these requirements, an abstract model of a multicopter is first provided. Then, some preliminaries on the state-norm-based DOC are provided. Finally, the objective of this Note is defined to find the optimal propulsion system that maximizes the DOC of the multicopter.

A. Abstract Modeling of a Multicopter

A multicopter hovering in the air is described by a linear dynamical model as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (1)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, and $\mathbf{u}(t) \in \Omega \subset \mathbb{R}^{n_p}$ is the control vector. Here, Ω is the control constraint set. In practice, the state $\mathbf{x}$ generally contains the information of the altitude and attitude, the lift of each propulsor, etc. Then, the definitions of the recovery region and the DOC are obtained according to the following [18–20]:

Definition 1: For system (1), the recovery region $\mathcal{R}$ within time t_r is defined as

$$\mathcal{R}(t_r) = \{\mathbf{x}(0) | \exists \mathbf{u}(t) \in \Omega, t \in [0, t_r], \text{ subject to } \mathbf{x}(t_r) = \mathbf{0}\} \quad (2)$$

Definition 2: For system (1), the DOC ρ_{doc} within time t_r is defined as

$$\rho_{\text{doc}} = \inf \|\mathbf{x}(0)\| \quad \forall \mathbf{x}(0) \notin \mathcal{R}(t_r) \quad (3)$$

where $\|\cdot\|$ represents the Euclidean norm.

From Definition 2, it is observed that the minimal distance from the origin to the boundary of the recovery region is considered to be the DOC of system (1). The value of ρ_{doc} represents how controllable the system is. A larger ρ_{doc} indicates a more controllable system,

which naturally leads to the question of how a multicopter could be designed to achieve a maximum ρ_{doc} .

B. Objective

To design a multicopter, a designer may have to select the battery packs, ESCs, motors, and propellers from a set of data sheets to assemble the multicopter. The data sheets are generally a collection of different propulsion systems, where each system is a group of battery packs, ESCs, motors, and propellers. Let n_p denote the number of propulsor physical parameters, which provide information about the battery packs, ESCs, motors, and propellers; and let $\boldsymbol{\eta}_i \in \mathbb{R}^{n_p}$ denote the parameter vector of the i th propulsor. Denote

$$\boldsymbol{\gamma} = [\boldsymbol{\eta}_1 \quad \boldsymbol{\eta}_2 \quad \cdots \quad \boldsymbol{\eta}_{n_d}]^T \in \mathbb{R}^{n_p \times n_d} \quad (4)$$

In practice, n_d is not required to be equal to n_p . For example, a traditional quadcopter has four identical propulsors; and one has $n_d = 1$ and $n_p = 4$. Then, the state matrices $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times n_p}$ in Eq. (1) are functions of $\boldsymbol{\gamma}$, namely, $\mathbf{A} = \mathbf{A}(\boldsymbol{\gamma})$ and $\mathbf{B} = \mathbf{B}(\boldsymbol{\gamma})$. The objectives of this Note are as follows:

Objective 1: Obtain the expressions of $\mathbf{A}(\boldsymbol{\gamma})$ and $\mathbf{B}(\boldsymbol{\gamma})$, based on which the state-norm-based DOC of a multicopter is computed.

Objective 2: Provide a methodology to optimize the propulsion system based on the state-norm-based DOC.

III. Solution to Objective 1

In this section, the dynamics of a complete multicopter system are derived, where the propulsor dynamics is considered and the relationships among $\mathbf{A}$, $\mathbf{B}$, and $\boldsymbol{\gamma}$ are obtained. As shown in Fig. 1, the complete multicopter system model contains the vehicle dynamics, propulsor dynamics, propeller model, motor model, and control effectiveness matrix.

A. Vehicle Dynamics

To obtain the linear model of a multicopter in the air, the following assumption is made for simplicity:

Assumption 1: All multicopters discussed here initially hover in the air, with no wind in the environment.

In practice, the aerodynamic damping and vehicle stiffness are ignorable if a multicopter is hovering. Then, the linear dynamical model of a multicopter around the hover condition ($p_z = p_{z0}$, and $v_z = \phi = \theta = \psi = 0$) is given as [1,21]

$$\dot{\boldsymbol{\xi}}_1 = \boldsymbol{\xi}_2 \quad \dot{\boldsymbol{\xi}}_2 = \mathbf{J}^{-1} \mathbf{B}_f \Delta \mathbf{f} \quad (5)$$

where $\boldsymbol{\xi}_1 = [\Delta p_z \quad \phi \quad \theta \quad \psi]^T$, $\boldsymbol{\xi}_2 = [v_z \quad \omega_x \quad \omega_y \quad \omega_z]^T$, $\Delta p_z = p_z - p_{z0}$, $\Delta \mathbf{f} = \mathbf{f} - \mathbf{f}_{ss}$, $\mathbf{J} = \text{diag}(-m_a, J_{xx}, J_{yy}, J_{zz}) \in \mathbb{R}^{4 \times 4}$, $\mathbf{g} = [g \quad 0 \quad 0 \quad 0]^T \in \mathbb{R}^4$, and $\mathbf{B}_f$ is the control effectiveness matrix. Here, $\mathbf{f} = [T_1 \quad \cdots \quad T_{n_p}]^T \in \mathbb{R}^{n_p}$, $\mathbf{f}_{ss} = [T_{ss,1} \quad \cdots \quad T_{ss,n_p}]^T$, and one has

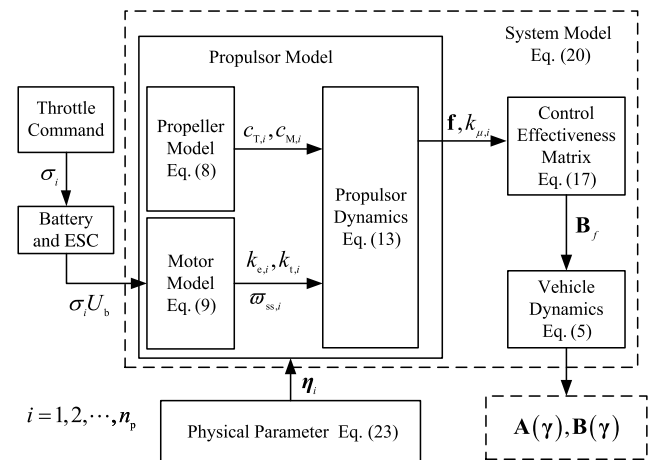


Fig. 1 Multicopter modeling process.

$$B_f \mathbf{f}_{ss} = m_a \mathbf{g} = \left[\sum_{i=1}^{i=n_p} T_{ss,i} \quad 0 \quad 0 \quad 0 \right]^T \quad (6)$$

when the multicopter is hovering and $\dot{v}_z = \dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$ in Ref. (5).

B. Propulsor Modeling

1. Propeller Modeling

Fixed-pitch propellers are often used for multicopters. When the multicopter is hovering and there is no wind, the thrust T_i and antitorque M_i of each propulsor i are given by

$$T_i = c_{T,i} \varpi_i^2, \quad M_i = c_{M,i} \varpi_i^2 \quad (7)$$

according to Ref. [5], where $c_{T,i}$ and $c_{M,i} \in \mathbb{R}_+$ are constant. According to Ref. [5], the constants $c_{T,i}$ and $c_{M,i}$ of propeller i are given by

$$c_{T,i} = \frac{1}{16} \pi \rho B_{p,i} \lambda \zeta^2 K_0 D_{p,i}^4 \frac{\varepsilon \arctan(H_{p,i}/\pi D_{p,i})}{\pi A + K_0},$$

$$c_{M,i} = \frac{1}{32A} \rho \zeta^2 B_{p,i}^2 C_d \lambda D_{p,i}^5 \quad (8)$$

The drag coefficient C_d is given by

$$C_d = C_{fd} + \frac{\pi A K_0^2 (\varepsilon \arctan(H_{p,i}/\pi D_{p,i}) - \alpha_0)^2}{e (\pi A + K_0)^2}$$

where C_{fd} is the zero-lift drag coefficient. In this Note, the standard air density of $\rho = 1.293 \text{ kg/m}^3$ is used; and the parameters $A, \varepsilon, \lambda, \zeta, e, C_{fd}, \alpha_0$, and K_0 are taken as

$$A = 5, \quad \varepsilon = 0.85, \quad \lambda = 0.75, \quad \zeta = 0.5, \quad e = 0.83,$$

$$C_{fd} = 0.015, \quad \alpha_0 = 0, \quad \text{and} \quad K_0 = 6.11$$

according to Ref. [5].

2. Motor Modeling

The ESC generates an equivalent average voltage $U_{e,i} = \sigma_i U_b$ after receiving the throttle command σ_i and the battery output voltage U_b . Then, the motor dynamics is given by

$$U_{m,i} = U_{e,i} - K_{e,i} \varpi_i \quad i_{m,i} = \frac{U_{m,i}}{R_{m,i}} \quad M_{m,i} = K_{t,i} i_{m,i}$$

$$\dot{\varpi}_i = \frac{1}{J_i} (M_{m,i} - M_i) \quad (9)$$

according to Ref. [6], where J_i can be computed using $m_{m,i}$ and $d_{m,i}$.

In practice, the parameters $K_{e,i}, K_{t,i}$ are computed by

$$K_{e,i} = \frac{U_{m0,i} - I_{m0,i} R_{m,i}}{K_{V0,i} U_{m0,i}}, \quad K_{t,i} = 9.55 K_{e,i} \quad (10)$$

where $U_{m0,i}, I_{m0,i}$, and $K_{V0,i}$ are given by the motor providers; and $U_{m0,i} = 10V$ is generally used in practice.

3. Propulsor Dynamics

According to Ref. [2] (p. 123), the model of propulsor i here is a complete power mechanism that includes not only a brushless dc motor but also an ESC and a propeller. Let us consider a hovering multicopter. Given the throttle command $\sigma_{ss,i}$, motor i will achieve a steady-state speed $\varpi_{ss,i}$. Then, the steady-state thrust $T_{ss,i}$ of propulsor i is

$$T_{ss,i} = c_{T,i} \varpi_{ss,i}^2 \quad (11)$$

Let $\varpi_i = \varpi_{ss,i} + \Delta \varpi_i$ and $\sigma_i = \sigma_{ss,i} + \Delta \sigma_i$, where $\Delta \varpi_i$ and $\Delta \sigma_i$ are small perturbations. Denote

$$\Delta \mathbf{f} = \mathbf{f} - \mathbf{f}_{ss} = [\Delta T_1 \quad \cdots \quad \Delta T_{n_p}]^T$$

$$\Delta \boldsymbol{\sigma} = [\Delta \sigma_1 \quad \cdots \quad \Delta \sigma_{n_p}]^T \quad (12)$$

Then, the propulsor dynamics is given by the following proposition:

Proposition 1: The propulsor dynamics of multicopters is expressed by

$$\Delta \dot{\mathbf{f}} = A_f \Delta \mathbf{f} + B_\sigma \Delta \boldsymbol{\sigma} \quad (13)$$

where

$$A_f = \begin{bmatrix} a_{f,1} & 0 & \cdots & 0 \\ 0 & a_{f,2} & \vdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & a_{f,n_p} \end{bmatrix}, \quad B_\sigma = \begin{bmatrix} b_{\sigma,1} \\ b_{\sigma,2} \\ \vdots \\ b_{\sigma,n_p} \end{bmatrix} \quad (14)$$

and

$$a_{f,i} = -\frac{K_{t,i} K_{e,i}}{J_i R_{m,i}} - \frac{2c_{M,i}}{J_i} \varpi_{ss,i}, \quad b_{\sigma,i} = 2c_{T,i} \varpi_{ss,i} \frac{K_{t,i} U_b}{J_i R_{m,i}} \quad (15)$$

Proof: The modeling processes are summarized in the Appendix. $\square$

According to the Appendix, given $\varpi_{ss,i}$, the throttle command $\sigma_{ss,i}$ is obtained by solving the following equation:

$$\frac{1}{J_i} \left(\frac{K_{t,i}}{R_{m,i}} (\sigma_{ss,i} U_b - K_{e,i} \varpi_{ss,i}) - c_{M,i} \varpi_{ss,i}^2 \right) = 0 \quad (16)$$

C. Control Effectiveness Matrix

From Eq. (5), the control effectiveness matrix B_f needs to be derived. According to Ref. [21], the control effectiveness matrix is computed by

$$B_f = \begin{bmatrix} 1 & \cdots & 1 \\ -r_1 \sin(\varphi_1) & \cdots & -r_{n_p} \sin(\varphi_{n_p}) \\ r_1 \cos(\varphi_1) & \cdots & r_{n_p} \cos(\varphi_{n_p}) \\ w_1 k_{\mu,1} & \cdots & w_{n_p} k_{\mu,n_p} \end{bmatrix} \quad (17)$$

where $r_i, i = 1, \dots, n_p$ is the distance from the center of the i th propulsor to the center of mass; $w_i, i = 1, \dots, n_p$ is defined by

$$w_i = \begin{cases} 1, & \text{if rotor } i \text{ rotates anticlockwise} \\ -1, & \text{if rotor } i \text{ rotates clockwise} \end{cases} \quad (18)$$

and $\varphi_i, i = 1, \dots, n_p$ is shown in Fig. 2. Here, $k_{\mu,i}, i = 1, \dots, n_p$ is computed by

$$k_{\mu,i} = \frac{M_i}{T_i} = \frac{c_{M,i}}{c_{T,i}} \quad (19)$$

according to Eq. (7).

D. System Model

Using Eqs. (5), (6), and (13), the multicopter model can be summarized as follows:

$$\dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u} \quad (20)$$

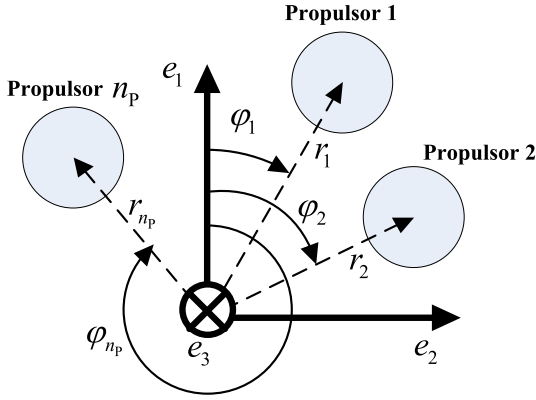


Fig. 2 Geometry of the multicopters [21].

where

$$\begin{aligned} \mathbf{x} &= [\Delta p_z \ \phi \ \theta \ \psi \ v_z \ \omega_x \ \omega_y \ \omega_z \ \Delta T_1 \ \Delta T_2 \ \cdots \ \Delta T_{n_p}]^T \\ &\in \mathbb{R}^{8+n_p} \\ \mathbf{u} &= [\Delta \sigma_1 \ \Delta \sigma_2 \ \cdots \ \Delta \sigma_{n_p}]^T \in \Omega \\ &= \{\mathbf{u} | -\sigma_{ss,i} \leq \Delta \sigma_i \leq 1 - \sigma_{ss,i}\} \end{aligned} \quad (21)$$

and

$$A = \begin{bmatrix} \mathbf{0}_{4 \times 4} & \mathbf{I}_4 & \mathbf{0}_{4 \times n_p} \\ \mathbf{0}_{4 \times 4} & \mathbf{0}_{4 \times 4} & \mathbf{J}^{-1} \mathbf{B}_f \\ \mathbf{0}_{n_p \times 4} & \mathbf{0}_{n_p \times 4} & \mathbf{A}_f \end{bmatrix}, \quad B = \begin{bmatrix} \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} \\ \mathbf{B}_\sigma \end{bmatrix} \quad (22)$$

From Eqs. (14), (15), (17), and (22), matrices A and B are functions of the battery parameter U_b , motor parameters ($K_{v0,i}$, $I_{m0,i}$, $R_{m,i}$, $m_{m,i}$, and $d_{m,i}$), and propeller parameters ($B_{p,i}$, $D_{p,i}$, and $H_{p,i}$). In the following, all these parameters are summarized by the vector $\boldsymbol{\eta}_i \in \mathbb{R}^9$, where

$$\boldsymbol{\eta}_i = [K_{v0,i} \ U_b \ m_{m,i} \ d_{m,i} \ I_{m0,i} \ R_{m,i} \ B_{p,i} \ D_{p,i} \ H_{p,i}]^T, \quad i = 1, 2, \dots, n_d \quad (23)$$

Then, $n_\gamma = 9$, $\boldsymbol{\gamma} = [\boldsymbol{\eta}_1 \ \boldsymbol{\eta}_2 \ \cdots \ \boldsymbol{\eta}_{n_d}]^T \in \mathbb{R}^{9 \times n_d}$ in Eq. (4), and $A(\boldsymbol{\gamma})$ and $B(\boldsymbol{\gamma})$ are given by Eq. (22). Moreover, the DOC of system (20) is a function of $\boldsymbol{\gamma}$ and is denoted by $\rho_{\text{doc}} = \rho_{\text{doc}}(\boldsymbol{\gamma})$.

E. Computation of Degree of Controllability

From the preceding, the multicopter model is given by Eq. (20) and the control constraint is given by Eq. (21). In Refs. [18,19], a DOC calculation method via system discretization is proposed to compute the DOC of system (20). According to Ref. [18], if the total recovery time t_r is divided into N equal intervals ΔT and if the control is restricted to be constant over each interval, then the state at the $(k+1)$ th step, in terms of the state at the k th step, is given by

$$\mathbf{x}_{k+1} = \mathbf{G}\mathbf{x}_k + \mathbf{H}\mathbf{u}_k \quad (24)$$

where $k = 0, 1, \dots, N-1$,

$$\mathbf{G} = e^{A\Delta T}, \quad \text{and} \quad \mathbf{H} = \int_0^{\Delta T} e^{-A\lambda} \mathbf{B} d\lambda$$

By substituting Eq. (24) into itself, one can obtain an expression for the final state of the system in terms of the initial state $\mathbf{x}_0$ and the discrete control sequence:

$$\mathbf{x}_N = \mathbf{G}^N \mathbf{x}(0) + \sum_{i=0}^{N-1} \mathbf{G}^{N-1-i} \mathbf{H} \mathbf{u}_i \quad (25)$$

Restricting $\mathbf{x}_N = 0$ in Eq. (25), one can obtain an expression for the set of all initial states $\mathbf{x}(0)$, which can be returned to the origin in N discrete steps:

$$\mathbf{x}(0) = \mathbf{M}\boldsymbol{\sigma} \quad (26)$$

where

$$\begin{aligned} \mathbf{M} &= -\mathbf{G}^{-N} [\mathbf{G}^{N-1} \mathbf{H} \ \mathbf{G}^{N-2} \mathbf{H} \ \cdots \ \mathbf{H}] \\ \boldsymbol{\sigma} &= [\mathbf{u}_0 \ \mathbf{u}_1 \ \cdots \ \mathbf{u}_{N-1}] \quad \mathbf{u}_i \in \Omega \end{aligned}$$

Then, the recovery region $\mathcal{R}$ defined in Definition 1 can be obtained from Eq. (26) by varying $\mathbf{u}_i$ in Ω . To obtain the DOC ρ_{doc} in Eq. (3), a toolbox was provided in Ref. [22].

Remark 1: According to Ref. [23], the maneuverability is roughly related to the feasible control input of an aircraft. The greater the maneuverability, the greater the margin that the feasible control can offer, roughly implying a larger DOC. Compared with the maneuverability, the DOC used in this Note is more exact because it has taken both the control input and the system dynamics into consideration. Then, the multicopter design method proposed in this Note can guarantee the maneuvering flight performances.

IV. Solution to Objective 2

In Sec. II, some preliminaries on the state-norm-based DOC are given. Then, the dynamics of a complete multicopter system are derived, wherein the propulsor dynamics is considered and the relationships among A , B , and $\boldsymbol{\gamma}$ are obtained in Sec. III. In practice, a set of data sheets is available to the designer for the various components, such as the battery packs, ESCs, motors, and propellers. Given the mission requirements (such as the endurance time and payload capacity), there may be many possible designs in the data sheets that can make the multicopter achieve the mission requirements. However, one may ask the following: “Which is the

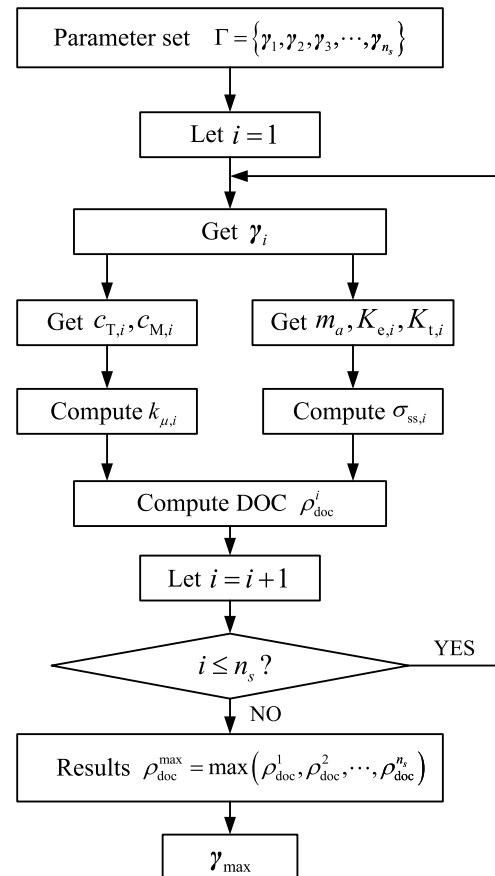


Fig. 3 Optimization process.

Algorithm 1 Optimized design process

- Step 1: Obtain the parameter set Γ from the data sheets for the various components, such as the battery packs, ESCs, motors, and propellers. Denote the element number of Γ as n_s and $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_{n_s}\}$.
- Step 2: Let $i = 1$.
- Step 3: Obtain the i th parameter vector γ_i .
- Step 4: Compute $c_{T,i}$ and $c_{M,i}$ according to Eq. (8), and $k_{\mu,i}$ is obtained by Eq. (19). Compute $K_{e,i}$ and $K_{t,i}$ according to Eq. (10).
- Step 5: Obtain the mass m_a of the multicopter based on γ_i . Then, the throttle command $\sigma_{ss,i}$ when the multicopter is hovering is obtained by solving Eqs. (11), (6), and (16).
- Step 6: Compute the DOC (which is denoted as ρ_{doc}^i) of the system in Eq. (20) based on the parameters $\gamma_i, c_{T,i}, c_{M,i}, k_{\mu,i}, K_{e,i}, K_{t,i}, m_a$, and $\sigma_{ss,i}$ using the toolbox provided by Ref. [22].
- Step 7: Let $i = i + 1$. If $i > n_s$, go to step 8. If $i \leq n_s$, go to step 3.
- Step 8: Obtain the maximum DOC $\rho_{\text{doc}}^{\max}$ among $\rho_{\text{doc}}^i, i = 1, 2, \dots, n_s$; and the corresponding γ is denoted as $\gamma_{\max}$. Then, $\gamma_{\max}$ are the desired multicopter parameters.

**Fig. 4 Coaxial hexacopter.**

optimal design among the various designs?" In this section, the DOC of the multicopter is used to obtain the optimal design among the various designs that satisfies the mission requirements.

When designing a multicopter, the data sheets will provide a set of γ , where $\gamma \in \Gamma \subset \mathbb{R}^{9 \times n_d}$. Given γ , one can obtain A and B based on

Eq. (22). Then, the DOC of system (20) is computed, and the design corresponding to the maximum DOC is considered as the optimal design. Therefore, the optimal multicopter design is obtained by solving the following optimization problem:

$$\max_{\gamma \in \Gamma} \rho_{\text{doc}}(\gamma) \quad (27)$$

subject to Eq. (20). Here, γ is the parameter vector that needs to be designed. The solution of optimization problem (27) is denoted as $\gamma_{\max}$ (indicating a group of battery packs, ESCs, motors, and propellers) corresponding to the maximum DOC (denoted as $\rho_{\text{doc}}^{\max}$).

The overall optimization process is depicted in Fig. 3. Then, a step-by-step process (called the optimized design process here) for the design of a multicopter based on DOC is summarized in Algorithm 1.

If there is no database, the optimal design can be obtained by solving the optimization problem expressed by Eq. (27). Then, shelf products or customized products are investigated, based on the optimal solutions. Existing methods can optimize the weight, flight time, efficiency, or payload capacity of a multicopter, whereas the method proposed in this Note can only optimize the DOC of a multicopter. In practice, one can use the existing methods to obtain a list of candidate designs of a multicopter, and then choose the design with the maximum DOC, based on the method proposed in this Note.

Table 1 Data sheets of motor and propeller parameters

IDs ^a	Motor parameters						Propeller parameters		
	$K_{V0,i}$, rpm/V	U_b , V	$d_{m,i}$, mm	$m_{m,i}$, g	$I_{m0,i}$, A	$R_{m,i}$, m Ω	D_p , in.	H_p , in.	B_p , g
1	2400	7.4	23	16	0.7	230	5	3	2
2	2400	7.4	23	16	0.7	230	6	2	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
312	100	44.4	86.8	400	1.4	95	29	9.5	2

^aDenotes identity document number for a group of battery packs, electronic speed controls, motors, and propellers.

Table 2 Optimal design results for a coaxial hexacopter

t_r , s	m_{vehicle} , kg	Surrounding propulsor					Middle propulsor					DOC
		ID	K_V , rpm/V	U_b , V	D_p , in.	H_p , in.	ID	K_V , rpm/V	U_b , V	D_p , in.	H_p , in.	
0.05	10	118	465	22.2	14	4.8	244	400	22.2	17	5.8	0.058
	12	118	465	22.2	14	4.8	300	170	44.4	22	6	0.058
	15	299	170	44.4	20	6	120	465	22.2	16	5.4	0.046
	17	300	170	44.4	22	6	279	420	22.2	16	5.4	0.039
	20	312	100	44.4	29	9.5	278	420	22.2	15	5	0.0309
0.1	10	118	465	22.2	14	4.8	244	400	22.2	17	5.8	0.299
	12	118	465	22.2	14	4.8	300	170	44.4	22	6	0.295
	15	299	170	44.4	20	6	120	465	22.2	16	5.4	0.204
	17	300	170	44.4	22	6	279	420	22.2	16	5.4	0.167
	20	312	100	44.4	29	9.5	278	420	22.2	15	5	0.131
0.5	10	118	465	22.2	14	4.8	244	400	22.2	17	5.8	128
	12	118	465	22.2	14	4.8	300	170	44.4	22	6	126
	15	299	170	44.4	20	6	120	465	22.2	16	5.4	13.1
	17	300	170	44.4	22	6	279	420	22.2	16	5.4	7.61
	20	312	100	44.4	29	9.5	278	420	22.2	15	5	5.41

Table 3 Different design results for a fixed vehicle mass

No.	m_{vehicle} , kg	Surrounding propulsor					Middle propulsor				
		ID	K_V , rpm/V	U_b , V	D_p , in.	H_p , in.	ID	K_V , rpm/V	U_b , V	D_p , in.	H_p , in.
1	10	118	465	22.2	14	4.8	244	400	22.2	17	5.8
2	10	126	400	22.2	17	5.8	206	830	14.8	12	4
3	10	307	100	29.6	28	9.2	223	760	14.8	14	4.8

Table 4 Optimal design results for a quadcopter

No.	t_r , s	m_{vehicle} , kg	ID	K_V , rpm/V	U_b , V	D_p , in.	H_p , in.	Hovering time, min	DOC
1	0.05	10	296	170	22.2	28	9.2	32.1	0.0072
2	0.05	10	293	135	44.4	22	6	27.5	0.0201
3	0.05	10	279	420	22.2	17	5.8	26	0.0263
4	0.05	10	274	280	22.2	22	6.6	28.2	0.0081
5	0.05	10	298	170	44.4	18	6	26.4	0.0451

V. An Example: Design of a Coaxial Hexacopter

In Sec. IV, an optimized design process for multicopter design based on DOC was provided. Here, the step-by-step process is used to design a coaxial hexacopter, as shown in Fig. 4. In practice, one can use middle coaxial propellers to provide constant thrust, whereas a quadcopter surrounding the middle propellers is used to control the altitude and attitude of the vehicle. However, one question arises: “Given the desired weight and size of the vehicle, how does one optimize the middle coaxial propellers and the surrounding quadcopter?”

A. Design Process and Results

A set of data sheets is provided by the suppliers, and there is a total of 312 pairs of motor–propeller groups in these data sheets (see Ref. [24] for the data sheets): namely, $n_s = 312$ in Algorithm 1. The key parameters of each motor–propeller group are shown in Table 1, where a part of the data sheets is listed. According to Assumption 1, the coaxial hexacopter is assumed to be hovering in the air without wind. The vehicle is viewed as a disk, and the diameter (denoted as d_{vehicle}) of this disk is constrained by $d_{\text{vehicle}} \leq 1.5$ m. In this section, five design targets are considered, where the desired masses (denoted as m_{vehicle}) of the vehicle are 10, 12, 15, 17, and 20 kg. According to Definition 1 and Definition 2, the recovery time t_r needs to be specified. In this Note, three cases of recovery times are considered, i.e., $t_r \in \{0.05 \text{ s}, 0.1 \text{ s}, 0.5 \text{ s}\}$. Given m_{vehicle} and t_r , the optimal motor–propeller group can be obtained according to Algorithm 1 in Sec. IV. Here, it is assumed that all the propulsors of the surrounding quadcopter are the same and that the two middle propulsors are the same. Then, the number of propulsors to be designed is two, namely, $n_d = 2$. The design results are shown in Table 2.

From Table 2, the following observations are obtained:

- 1) For a fixed m_{vehicle} , the recovery time t_r does not affect the optimal design results.
- 2) The DOC of the multicopter decreases as m_{vehicle} increases.

B. Discussions

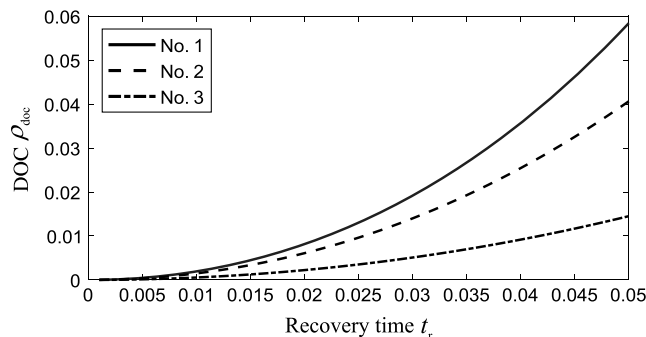
Discussions of the results based on the example in Sec. V are presented in the following to demonstrate the effectiveness of the proposed design methodology.

1) For a fixed m_{vehicle} , the recovery time t_r does not affect the optimal design results; namely, we will obtain the same optimal design regardless of the value of t_r . Given $t_r = t_r^*$, if the DOC of design a is larger than that of design b, then the DOC of design a is always larger than that of design b, regardless of the value of t_r . This fact is further shown in Fig. 5, where the DOCs of different design results (see Table 3) for $m_{\text{vehicle}} = 10$ kg are computed, and where the recovery time t_r varies from 0.001 to 0.05 s. It is shown that the DOC of design 1 (listed in Table 3) is always larger than the DOC of design 2 and design 3. Then, there is no need to set a large t_r in practice, because this will require excessive computing resources and not provide a benefit.

2) The DOC of the multicopter decreases as m_{vehicle} increases; namely, it is more difficult to control a heavier multicopter than a lighter one. For a fixed m_{vehicle} , as shown in Table 3 and Fig. 5, design 1 has the largest DOC where the surrounding quadcopter has smaller propellers than the middle propulsors. These facts are consistent with the results in Ref. [2] (p. 13): a multicopter with a larger mass has a larger moment of inertia and requires larger propulsors, which have slower dynamics. Thus, it is more difficult to change its state (such as position and attitude) than for a smaller multicopter.

VI. Further Discussions

In practice, the method proposed in this Note is usually integrated with the traditional multicopter design process to reduce the iterations. To provide additional results, we use a multicopter design tool[‡] to design a quadcopter (with a total weight of 10 kg and a hovering time of 20 min). Five recommended configurations are provided, and the results are shown in Table 4. However, configuration 1 has the longest hovering time but the smallest DOC; namely, it is difficult to change the state of the quadcopter under configuration 1. Based on the idea of this Note, it is better to choose configuration 5 to obtain more control capacity, and the design requirements (a total weight of 10 kg and a hovering time of 20 min) are satisfied.

**Fig. 5** DOCs of different design results for $m_{\text{vehicle}} = 10$ kg.

[‡]Data available online at <http://www.flyeval.com/recalc.html> [retrieved 02 May 2019].

VII. Conclusions

The optimal design problem of a multicopter is considered in this Note. First, a complete multicopter system model is derived, wherein the propulsor dynamics is considered. Then, based on the complete multicopter model, an optimized design methodology for multicopters based on the degree of controllability is proposed. In addition, a step-by-step process is derived from the methodology, which is used to design a coaxial hexacopter with different desired vehicle masses to demonstrate its effectiveness. Further discussions about the design results of the coaxial hexacopter are presented to demonstrate that the design methodology is effective.

Appendix: Proof of Proposition 1

From Eqs. (7) and (9), the propulsor dynamic model is equivalent to the motor-propeller dynamic model, which is given as

$$\ddot{\omega}_i = f_{\omega}(\omega_i, \sigma_i) = \frac{1}{J_i} \left(\frac{K_{t,i}}{R_{m,i}} (\sigma_i U_b - K_{e,i} \omega_i) - c_{M,i} \omega_i^2 \right) \quad (\text{A1})$$

Let us consider a hovering multicopter. Given the throttle command $\sigma_{ss,i}$, the motor can achieve a steady-state speed $\omega_{ss,i}$, which can be obtained by solving the following equation:

$$\frac{1}{J_i} \left(\frac{K_{t,i}}{R_{m,i}} (\sigma_{ss,i} U_b - K_{e,i} \omega_{ss,i}) - c_{M,i} \omega_{ss,i}^2 \right) = 0 \quad (\text{A2})$$

Then, one has

$$f_{\omega}(\omega_{ss,i}, \sigma_{ss,i}) \equiv 0 \quad (\text{A3})$$

Let $\omega_i = \omega_{ss,i} + \Delta\omega_i$ and $\sigma_i = \sigma_{ss,i} + \Delta\sigma_i$, where $\Delta\omega_i$ and $\Delta\sigma_i$ are small perturbations; then,

$$\begin{aligned} \Delta\dot{\omega}_i &= f_{\Delta\omega}(\Delta\omega_i, \Delta\sigma_i) \\ &= \frac{1}{J_i} \left(\frac{K_{t,i}}{R_{m,i}} ((\sigma_{ss,i} + \Delta\sigma_i) U_b - K_{e,i} (\omega_{ss,i} + \Delta\omega_i)) \right. \\ &\quad \left. - c_{M,i} (\omega_{ss,i} + \Delta\omega_i)^2 \right) \end{aligned} \quad (\text{A4})$$

according to Eq. (A1). From Eqs. (A1), (A3), and (A4), an equilibrium state $\Delta\omega_i = 0$, $\Delta\sigma_i = 0$ is obtained for system (A4). Then, one has $f_{\Delta\omega}(0, 0) = 0$.

To design the multicopter based on the DOC, the linear propulsor dynamic model is required. According to Ref. [25] (p. 53), a nonlinear system should behave similarly to its linearized approximation for small range motions; then, the linear approximation of system (A4) at $(\Delta\omega_i = 0, \Delta\sigma_i = 0)$ is given by

$$\Delta\dot{\omega}_i = a_{\omega,i} \Delta\omega_i + b_{\omega,i} \Delta\sigma_i$$

where

$$\begin{aligned} a_{\omega,i} &= \left. \frac{\partial f_{\Delta\omega}(\Delta\omega_i, \Delta\sigma_i)}{\partial \Delta\omega_i} \right|_{\Delta\omega_i=0, \Delta\sigma_i=0} = -\frac{K_{t,i} K_{e,i}}{J_i R_{m,i}} - \frac{2c_{M,i}}{J_i} \omega_{ss,i} \\ b_{\omega,i} &= \left. \frac{\partial f_{\Delta\omega}(\Delta\omega_i, \Delta\sigma_i)}{\partial \Delta\sigma_i} \right|_{\Delta\omega_i=0, \Delta\sigma_i=0} = \frac{K_{t,i} U_b}{J_i R_{m,i}} \end{aligned}$$

From Eq. (7), one has

$$T_i = c_{T,i} (\omega_{ss,i} + \Delta\omega_i)^2 \approx T_{ss,i} + 2c_{T,i} \omega_{ss,i} \Delta\omega_i$$

as $\Delta\omega_i \rightarrow 0$. Let

$$\Delta T_i = T_i - T_{ss,i} \triangleq 2c_{T,i} \omega_{ss,i} \Delta\omega_i$$

Then,

$$\begin{aligned} \Delta\dot{T}_i &= 2c_{T,i} \omega_{ss,i} \Delta\dot{\omega}_i \\ &= 2c_{T,i} \omega_{ss,i} a_{\omega,i} \Delta\omega_i + 2c_{T,i} \omega_{ss,i} b_{\omega,i} \Delta\sigma_i \\ &= a_{f,i} \Delta T_i + b_{\sigma,i} \Delta\sigma_i \end{aligned} \quad (\text{A5})$$

where

$$a_{f,i} = a_{\omega,i}, \quad b_{\sigma,i} = 2c_{T,i} \omega_{ss,i} b_{\omega,i} \quad (\text{A6})$$

From Eqs. (12) and (A5), one has the propulsor dynamics shown in Eq. (13).

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