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ABSTRACT

There exist at least two viewpoints on the internal model principle (IMP), namely the cancelation viewpoint and the geometrical viewpoint. However, neither of them is applicable to repetitive control (RC, or repetitive controller, also designated RC) of nonlinear systems directly. Because of this, error dynamics are often derived to transform a periodic signal tracking problem into a rejection problem. This not only fails to represent the special feature of periodic signals but also restricts the applications of RC. In view of this, this paper proposes a new viewpoint on IMP, namely the actuator-focussed viewpoint. With this, the periodic signal tracking problem can be converted into a stability problem without deriving error dynamics for linear periodic systems and nonlinear systems. In order to demonstrate its effectiveness, the proposed design approach is applied to RC problems for a linear periodic system, a minimum-phase nonlinear system and a nonminimum-phase nonlinear system.

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Repetitive control; nonlinear systems; periodic systems; internal model principle; non-minimum phase

1. Introduction

The *internal model principle* (IMP) was first proposed by Francis and Wonham (1976) and Wonham (1976). It states that *if any exogenous signal can be regarded as the output of an autonomous system, then the inclusion of this signal model, namely internal model, in a stable closed-loop system can assure asymptotic tracking or asymptotic rejection of the signal*. Until now, to the best of the authors' knowledge, there exist at least two viewpoints on IMP. In the early years, for linear time-invariant (LTI) systems, IMP implies that the internal model is to supply closed-loop transmission zeros which cancel the unstable poles of the disturbances and reference signals. This is called *cancelation viewpoint* here and only works for problems able to be formulated in terms of transfer functions. In the mid-1970s, Francis and Wonham proposed the geometric approach (Wonham, 1979) to design an internal model controller (Francis & Wonham, 1976; Wonham, 1976). The purpose of internal models is to construct an invariant subspace for the closed-loop system and make the regulated output zero at each point of the invariant subspace. This is called *geometrical viewpoint* here. By the geometrical viewpoint, Isidori and Byrnes in the early 1990s further extended it from LTI systems to nonlinear time-invariant systems (Isidori & Byrnes, 1990). This work inspires the development of internal-model-based controller design methods greatly up to now (Chen & Huang, 2014; Huang, 2004; Isidori, Marconi, & Serrani, 2003; Knobloch, Isidori, & Flockerzi, 2014; Memon & Khalil, 2010; Trip, Burger, & De Persis, 2016; Wieland, Sepulchre, & Allgower, 2011). Also, by the geometrical viewpoint, the finite-dimensional output regulation problem has been generalised for both infinite-dimensional systems and reference/disturbance signals generated by some infinite-dimensional exosystems (Byrnes, Laukó, Gilliam, & Shubov, 2000; Hämäläinen & Pohjolainen, 2010; Hara, Yamamoto, Omata, & Nakano, 1988; Immonen, 2007a, 2007b; Natarajan, Gilliam, & Weiss, 2014; Paunonen & Pohjolainen, 2010; Weiss & Häfele, 1999; Xu & Dubljevic, 2017). However, the two viewpoints on IMP are difficult to handle general periodic signal tracking problems for nonlinear systems subject to periodic disturbances generated by infinite-dimensional exosystems, because the resulting closed-loop system, which contains the copy of such an exosystem, is nonlinear and infinite-dimensional. On the one hand, the cancelation viewpoint cannot be applied to nonlinear systems directly because it relies on transfer functions. On the other hand, the existing theories on geometric approach for infinite-dimensional systems cannot be applied to nonlinear systems directly because they rely on the linear operator theory.

Any T-periodic signal can be regarded as the output of an autonomous system model $\frac{1}{1-e^{-sT}}$, an infinite-dimensional model. A controller including the internal model $\frac{1}{1-e^{-sT}}$ is called a *repetitive controller* (RC, or *repetitive control*, also designated RC), and a system including such a controller is referred to as an RC system. The basic idea of RC stems from the cancelation viewpoint on the IMP. RC initially developed for continuous single-input, single-output (SISO) LTI systems by Inoue et al., for high accuracy tracking of a periodic signal with a known period. Later, Hara et al. extended the RC to multiple-input, multiple-output (MIMO) LTI systems (Hara et al., 1988). Since then, RC has begun to receive more attention and applications, and has become a special topic in control theory. In recent years, the development on RC has been uneven. By the use of frequency-domain methods, the theories and applications of LTI systems have developed very well (Longman, 2000; Rogers & Owens, 1992). On the other hand, RC in nonlinear systems

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has received insufficient research effort (Quan & Cai, 2010). Currently, there exist two major ways to design RCs for nonlinear systems. One way is to transform a nonlinear system into an LTI system with/without a weak nonlinear term, then apply existing design methods of LTI systems to the transformed system (Alleyne & Pomykalski, 2000; Ghosh & Paden, 2004; Lee & Tsao, 2004; Lin, Chung, & Hung, 1991; Ma, 1990). However, not all nonlinear systems can be transformed into simple forms. What is more, in these early literature, only the stability of the closed-loop with an RC internal model was considered. It was supposed that the RC could attenuate the periodic component at the tracking error for nonlinear systems as for LTI systems. But, the reasons were not found. In fact, the proposed new viewpoint on the IMP can be used to explain the reason. The other way is to transform a tracking problem for nonlinear systems to a rejection problem for nonlinear error dynamics, then apply existing adaptive-control-like methods to the transformed error dynamics (Chien & Tayebi, 2008; Dixon, Zergeroglu, & Costic, 2002; Kim & Ha, 2000; Messner, Horowitz, Kao, & Boals, 1991; Quan & Cai, 2011; Sadegh, Horowitz, Kao, & Tomizuka, 1990; Sun, Ge, & Mareels, 2006; Xu & Yan, 2006). Currently, the adaptive-control-like method is the leading method of designing RCs in nonlinear systems. Compared with the internal-model-based controller design, the structures of RCs obtained are similar or the same, but the ways to obtain them are very different. By the internal-model-based controller design, error dynamics do not need to be derived. However, by the adaptive-control-like method, error dynamics are derived to convert a tracking problem to a rejection problem, during which full desired states are required. This not only fails to represent the special feature of T-periodic signals but also restricts the application of RC. For nonminimum-phase nonlinear systems, the ideal internal dynamics are required to obtain the error dynamics. This is difficult and computationally expensive especially when the internal dynamics are subject to an unknown disturbance (Shkolnikov & Shtessel, 2002). As a result, the authors suppose that this is the reason why few RC works on nonminimum-phase nonlinear systems have been reported.

Based on the consideration above, this paper proposes a new viewpoint on the IMP, namely the *actuator-focussed viewpoint*. The actuator-focussed viewpoint can overcome the drawbacks mentioned in both the cancelation viewpoint and the geometrical viewpoint when general periodic external signals are considered. By this proposed new viewpoint, the actuator-focussed RC design method is further proposed for T-periodic signal tracking of linear periodic systems and nonlinear systems. The major contributions of this paper are: (i) the new actuator-focussed viewpoint on the IMP which overcomes some drawbacks in both the cancelation viewpoint and the geometrical viewpoint; (ii) the actuator-focussed RC design method which can unify the RC design for LTI systems, linear periodic systems, and nonlinear systems; (iii) the proposed method facilitating the controller design and simplifying the designed controllers.

The following notation is used. $\mathbb{R}^n$ is Euclidean space of dimension n and $\mathbb{N}$ denotes nonnegative integers. $\|\cdot\|$ denotes the Euclidean norm or a matrix norm induced by the Euclidean norm. $\mathcal{C}([a,b],\mathbb{R}^n)$ denotes the space of continuous n-dimension function vector on $[a,b]$. The symbol $\mathbf{x}_t \in \mathcal{C}([a,b],\mathbb{R}^n)$ implies $\mathbf{x}_t(\theta) \triangleq \mathbf{x}(t+\theta)$, $\theta \in [a,b]$. $\mathcal{C}^n_{PT}([0,\infty);\mathbb{R}^m)$ is the space of nth-order continuously differentiable functions $\mathbf{f}:[0,\infty)\to\mathbb{R}^m$ which are T-periodic, i.e. $\mathbf{f}(t+T)=\mathbf{f}(t)$. If $\mathbf{x}(t)$ is bounded on $[0,\infty)$, let $\|\cdot\|_a$ denote the quantity $\|\mathbf{x}\|_a \triangleq \limsup_{t\to\infty}\|\mathbf{x}(t)\|$ (Teel, 1996). The font of a scalar is normal, while the corresponding vector symbol is bold font. For example, e,v are scalars, while $\mathbf{e}$, $\mathbf{v}$ are vectors.

The remainder of this paper is organised as follows. Section 2 introduces some preliminaries. In Section 3, the actuator-focussed viewpoint on IMP is presented, where two examples about the step signal and general T-periodic signal show the general idea. By the aforementioned actuator-focussed viewpoint, the actuator-focussed RC design method is further proposed in Section 4 to establish conditions that the viewpoint requires. In Section 5, the actuator-focussed RC design method is further proposed to solve three periodic signal tracking problems. Finally, conclusions are given in Section 6.

2. Preliminaries

Consider a general perturbed time-delay system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}_t, \mathbf{w}), \quad t \geq t_0 \tag{1}$$

with $\mathbf{x}_{t_0}(\theta) = \boldsymbol{\phi}(\theta)$, $\theta \in [-T, 0]$, $T > 0$, where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{w}(t) \in \mathbb{R}^m$ is a piecewise continuous and bounded perturbation. The function $\mathbf{f}:[t_0,\infty)\times\mathcal{C}([-T,0],\mathbb{R}^n)\times\mathbb{R}^m\to\mathbb{R}^n$ is supposed to be continuous and takes bounded sets into bounded sets. Here, let initial time $t_0 = 0$ for simplicity.

Definition 2.1: The solutions $\mathbf{x}_t(\boldsymbol{\phi})$ of system (1) with $\mathbf{x}_{t_0}(\theta) = \boldsymbol{\phi}(\theta)$, $\theta \in [-T, 0]$ are said to be *uniformly bounded*, if for each $\delta > 0$ there exists $\varepsilon > 0$ such that $\|\mathbf{x}(\boldsymbol{\phi})(t)\| \leq \varepsilon$, $t \geq t_0$, when $\sup_{\theta\in[-T,0]} \|\boldsymbol{\phi}(\theta)\| < \delta$.

Definition 2.2: The solutions $\mathbf{x}_t(\boldsymbol{\phi})$ of system (1) with $\mathbf{x}_{t_0}(\theta) = \boldsymbol{\phi}(\theta)$, $\theta \in [-T, 0]$ are said to be *uniformly ultimately bounded* with *ultimate bound* ε, if for each $\delta > 0$ there exists $T_1 = T_1(\varepsilon, \delta) > 0$ independent of t_0 such that $\|\mathbf{x}(\boldsymbol{\phi})(t)\| \leq \varepsilon$ for all $t \geq t_0 + T_1$ when $\sup_{\theta\in[-T,0]} \|\boldsymbol{\phi}(t)\| < \delta$.

Lemma 2.1 (Burton, 1985, pp. 249–251): *Suppose* (i) $\mathbf{f}(t, \mathbf{x}_t, \mathbf{w}(t)) = \mathbf{f}(t + T, \mathbf{x}_t, \mathbf{w}(t + T))$, (ii) $\mathbf{f}(t, \mathbf{x}_t, \mathbf{w})$ *satisfies a local Lipschitz condition in* $\mathbf{x}_t$, (iii) $\mathbf{x}(t + T)$ *is a solution of* (1) *whenever* $\mathbf{x}(t)$ *is a solution of* (1). *If solutions of* (1) *are uniformly bounded and uniformly ultimately bounded, then* (1) *has a T-periodic solution.*

3. Actuator-focussed viewpoint on IMP

The general idea of the actuator-focussed viewpoint on IMP is introduced first. Then it is used to explain the role of the internal models for step signals and general periodic signals, respectively. Finally, the viewpoint is extended to filtered repetitive control (FRC, or filtered repetitive controller, also designated FRC) systems. This section is only to clarify the actuator-focussed viewpoint in the aspect of signals. The next section will introduce how to generate such signals required by designing controllers.

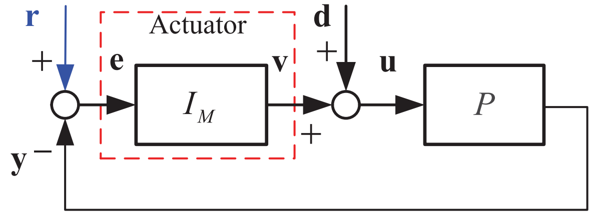

Figure 1. A general system including the internal model.

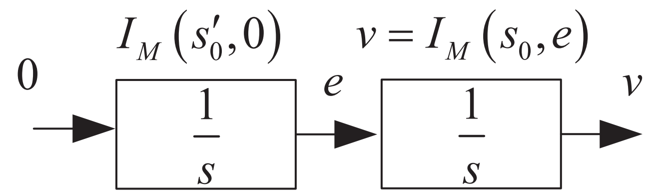

Figure 2. The composition of two integral terms.

3.1 General idea

As shown in Figure 1, some definitions are given to clarify the general idea of the actuator-focussed viewpoint. The internal model is defined as

$$\mathbf{v} = I_M(\mathbf{s}_0, \mathbf{e}). \tag{2}$$

Here $I_M : \mathcal{S}_0 \oplus \mathcal{E} \to \mathcal{V}$, where $\mathcal{S}_0$ is a Banach space for the initial conditions (the initial condition can have many forms, such as $\mathbf{s}_0 \in \mathbb{R}^n$ or $\mathbf{s}_0(\theta) = \boldsymbol{\phi}(\theta) \in \mathbb{R}^n$, $\theta \in [-T, 0]$); $\mathbf{e}(t) \in \mathcal{E} = \{f : \mathbb{R} \to \mathbb{R}^m\}$ and $\mathbf{v}(t) \in \mathcal{V} = \{f : \mathbb{R} \to \mathbb{R}^m\}$ are signal spaces, representing the input and output of the internal model, respectively. Let $\mathcal{I}_M = \{\mathbf{s} \in \mathcal{V} | \mathbf{s} = I_M(\mathbf{s}_0, \mathbf{0}),\ \forall\ \mathbf{s}_0 \in \mathcal{S}_0\}$, which denotes the set of all possible outputs. For example, $\mathcal{I}_M$ can be a set of step signals with different amplitudes (Section 3.2.1), or a set of sinusoids at frequency ω with different phases, or a set of T-periodic signals with different shapes (Section 3.2.2). For any $\mathbf{s}_1 \in \mathcal{I}_M$, there exists an initial condition $\mathbf{s}_{10} \in \mathcal{S}_0$ such that $\mathbf{s}_1 = I_M(\mathbf{s}_{10}, \mathbf{0})$. Similarly, as shown in Figure 1, the plant P is defined as

$$\mathbf{y} = P(\mathbf{x}_0, \mathbf{u}).$$

Here $P : \mathcal{X}_0 \oplus \mathcal{U} \to \mathcal{Y}$, where $\mathcal{X}_0$ is a Banach space for the initial conditions; $\mathbf{u}(t) \in \mathcal{U} = \{f : \mathbb{R} \to \mathbb{R}^m\}$ and $\mathbf{y}(t) \in \mathcal{Y} = \{f : \mathbb{R} \to \mathbb{R}^m\}$ are signal spaces, representing the input and output of the plant, respectively. The *steady states* $\mathbf{e}^*$ and $\mathbf{v}^*$ of the general system shown in Figure 1 imply that there exist $\mathbf{s}_0^* \in \mathcal{S}_0$ and $\mathbf{x}_0^* \in \mathcal{X}_0$ such that

$$\begin{aligned} \mathbf{e}^* &= \mathbf{r} - P\left(\mathbf{x}_0^*, \mathbf{d} + \mathbf{v}^*\right) \\ \mathbf{v}^* &= I_M\left(\mathbf{s}_0^*, \mathbf{e}^*\right). \end{aligned} \tag{3}$$

Example 3.1 (on the composition of internal models): Generally, the internal model I_M is marginally stable so that it can generate non-vanishing and bounded signals with any nonzero initial condition. However, $I_M(\mathbf{s}_0, I_M(\mathbf{s}_0', \mathbf{0}))$ is often *unbounded* if $\mathbf{s}_0' \neq \mathbf{0}$ no matter what $\mathbf{s}_0$ is. As shown in Figure 2, if I_M is an integral term whose Laplace transform is $1/s$, then

$$I_M\left(s_0', 0\right) = s_0' \mathcal{L}^{-1}\left(\frac{1}{s}\right) = s_0' u(t)$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform, $u(t)$ is the unit step function, and $s_0' \in \mathbb{R}$ is the initial condition. However, it can be observed that

$$\begin{aligned} I_M\left(s_0, I_M\left(s_0', 0\right)\right) &= s_0 \mathcal{L}^{-1}\left(\frac{1}{s}\right) + s_0' \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) \\ &= s_0 u(t) + s_0' t u(t). \end{aligned}$$

The signal $I_M(s_0, I_M(s_0', 0))$ is unbounded when $s_0' \neq 0$ no matter what s_0 is. Therefore, the compound $I_M(s_0, I_M(s_0', 0)) \notin \mathcal{I}_M$ if $s_0' \neq 0$. If I_M is an internal model about sinusoids at frequency ω whose Laplace transform is $1/(s^2 + \omega^2)$, then

$$I_M\left(s_0', 0\right) = s_0' \mathcal{L}^{-1}\left(\frac{1}{s^2 + \omega^2}\right) = s_0' \sin(\omega t)$$

where $s_0' \in \mathbb{R}$ is the initial condition. Also, it can be shown that $I_M(s_0, I_M(s_0', 0))$ is unbounded when $s_0' \neq 0$ no matter what s_0 is. Therefore, the compound $I_M(s_0, I_M(s_0', 0)) \notin \mathcal{I}_M$ if $s_0' \neq 0$.

Theorem 3.1: *As shown in Figure* 1, *suppose that*

(i) *The composition* $I_M(\mathbf{s}_0, I_M(\mathbf{s}_0', \mathbf{0})) \notin \mathcal{I}_M$ *if* $\mathbf{s}_0' \neq \mathbf{0}$;
(ii) *The steady states* $\mathbf{e}^*, \mathbf{v}^* \in \mathcal{I}_M$;
(iii) $I_M(\mathbf{s}_0, \mathbf{0}) \equiv \mathbf{0}$ *if and only if* $\mathbf{s}_0 = \mathbf{0}$.

Then $\mathbf{e}^* = \mathbf{0}$.

Proof: Prove it by contradiction, namely $\mathbf{e}^* \neq \mathbf{0}$. Since $\mathbf{e}^* \in \mathcal{I}_M$ by condition (ii), there exists an initial condition $\mathbf{s}_0'$ such that $\mathbf{e}^* = I_M(\mathbf{s}_0', \mathbf{0})$ according to definition of $\mathcal{I}_M$. Then, since $\mathbf{e}^* \neq \mathbf{0}$ as supposed, $\mathbf{s}_0' \neq \mathbf{0}$ according to condition (iii). Furthermore, according to (3), one further has

$$\mathbf{v}^* = I_M\left(\mathbf{s}_0^*, \mathbf{e}^*\right) = I_M\left(\mathbf{s}_0^*, I_M\left(\mathbf{s}_0', \mathbf{0}\right)\right).$$

Since $\mathbf{s}_0' \neq \mathbf{0}$, $\mathbf{v}^* = I_M(\mathbf{s}_0^*, I_M(\mathbf{s}_0', \mathbf{0})) \notin \mathcal{I}_M$ according to condition (i). This contradicts with $\mathbf{v}^* \in \mathcal{I}_M$ in condition (ii). So, $\mathbf{e}^* = \mathbf{0}$. ■

Remark 3.1: For condition (i) in Theorem 3.1, it is reasonable as shown in Example 3.1. Also, it is necessary. The form $I_M(\mathbf{s}_0, I_M(\mathbf{s}_0', \mathbf{0}))$ does not imply $I_M(\mathbf{s}_0, I_M(\mathbf{s}_0', \mathbf{0})) \notin \mathcal{I}_M$ according to the definition, because there may exist $\mathbf{s}_0'' \in \mathcal{S}_0$ such that $I_M(\mathbf{s}_0, I_M(\mathbf{s}_0', \mathbf{0})) = I_M(\mathbf{s}_0'', \mathbf{0})$. Condition (ii) in Theorem 3.1 implies states $\mathbf{e}, \mathbf{v}$ will tend to steady states belonging to $\mathcal{I}_M$ eventually. Condition (iii) is also reasonable as shown in Example 3.1. In the following Sections 3.2.1 and 3.2.2, Theorem 3.1 will be applied to two examples. Readers can obtain more intuitional explanation.

Remark 3.2: In the analysis, the controller (actuator), namely (2) or see the dashed box in Figure 1, is focussed on. For this reason, the new viewpoint is called the *actuator-focussed viewpoint*.

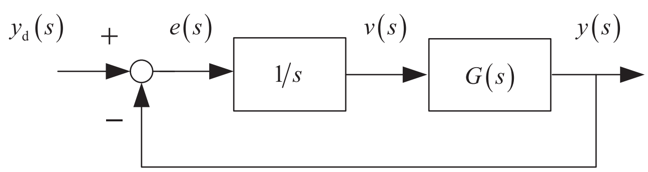

Figure 3. Step signal tracking.

3.2 Two examples

Two examples will further show the general idea on the actuator-focussed viewpoint in detail.

3.2.1 Step signal

Since the Laplace transformation model of a unit step signal and an integral term are the same, namely $1/s$, the inclusion of the model $1/s$ in a stable closed-loop system can assure perfect tracking or complete rejection of the unit step signal according to IMP.

Cancelation Viewpoint. As shown in Figure 3, the transfer function from the desired signal to the tracking error is written as follows

$$\begin{aligned} e(s) &= \frac{1}{1+\frac{1}{s}G(s)} y_d(s) = \frac{1}{s+G(s)}\left(s\frac{1}{s}\right) \\ &= \frac{1}{s+G(s)}. \end{aligned} \tag{4}$$

Then it only requires to verify whether or not the roots of the equation $s + G(s) = 0$ are all in the left s-plane, namely whether or not the closed-loop system is stable. If all roots are in the left s-plane, then the tracking error tends to zero as $t \to \infty$. Therefore, the tracking problem has been reduced to a stability problem of the closed-loop system.

Actuator-Focused Viewpoint. The actuator-focussed viewpoint in the following will give a new explanation on IMP without using transfer functions. In this example, I_M is an integral term whose Laplace transform is $1/s$. The conditions (i) and (iii) of Theorem 3.1 hold as shown in Example 3.1. In the following, the condition (ii) of Theorem 3.1 is examined. Let $G(s) = \mathbf{c}^{\mathrm{T}}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, $d \in \mathbb{R}$. The minimal realisation of $y = G(s)v$ is

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{b}v \\ y &= \mathbf{c}^{\mathrm{T}}\mathbf{x} + dv. \end{aligned}$$

As shown in Figure 3, the resulting closed-loop system becomes

$$\underbrace{\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{v} \end{bmatrix}}_{\dot{\mathbf{z}}} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{b} \\ -\mathbf{c}^{\mathrm{T}} & -d \end{bmatrix}}_{\mathbf{A}_a} \underbrace{\begin{bmatrix} \mathbf{x} \\ v \end{bmatrix}}_{\mathbf{z}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ y_d \end{bmatrix}}_{\mathbf{w}}. \tag{5}$$

The solution is

$$\mathbf{z}(t) = e^{\mathbf{A}_a t}\mathbf{z}(0) + \int_0^t e^{\mathbf{A}_a(t-s)}\mathbf{w}\,ds$$

where $\mathbf{w}$ is constant. If the closed-loop system is stable, then the matrix $\mathbf{A}_a$ is stable, namely the real parts of eigenvalues of $\mathbf{A}_a$ are negative. As a result, $\mathbf{z}(t)$ will tend to a constant vector, say $[\mathbf{x}^{*\mathrm{T}}\ v^*]^{\mathrm{T}}$, as $t \to \infty$. Consequently, $v(t)$ and $e(t) = y_d - \mathbf{c}^{\mathrm{T}}\mathbf{x}(t) - dv(t)$ will tend to constants v^* and $e^* = y_d - \mathbf{c}^{\mathrm{T}}\mathbf{x}^* - dv^*$ as $t \to \infty$, respectively. Therefore, $v^*, e^* \in \mathcal{I}_M$. According to Theorem 3.1, it can be claimed that $e^* = 0$. An intuitive interpretation is also given by contradiction. One has

$$\dot{v}(t) = e(t) \tag{6}$$

because of the integral term. If $e^* \neq 0$, then $v(t) = \int e(s)\,ds$ will tend to infinity as $t \to \infty$. This contradicts with v^* being a constant. So, $e^* = 0$. The explanation is somewhat different from Theorem 3.1, but their essential ideas are the same. As shown above, to confirm that the tracking error $e(t) \to 0$ as $t \to \infty$, it is only required to verify whether or not the closed-loop system without external signals is exponentially stable. This implies that the tracking problem has been reduced to a stability problem.

Remark 3.3: Another way is given to explain why $v(t)$ and $e(t)$ will tend to constants as $t \to \infty$ in the following. If the closed-loop system without external signals is exponentially stable, then when the system is driven by a unit step signal, the closed-loop system is uniformly bounded and uniformly ultimately bounded. By the *fixed point theory* (Burton, 1985, pp. 164–182), there exists a constant solution to (5). Since the closed-loop system without external signals is exponentially stable, $v(t)$ and $e(t)$ will tend to the constant solution as $t \to \infty$.

3.2.2 General T-periodic signal

If the external signal is in the form of $y_d(t) = y_d(t - T)$, which can represent any T-periodic signal, then perfect tracking or complete rejection can be achieved by incorporating the model $1/(1 - e^{-sT})$ into the closed-loop system.

Cancelation Viewpoint. Similarly, as shown in Figure 4, the transfer function from the desired signal to the error is written as follows

$$\begin{aligned} e(s) &= \frac{1}{1+\frac{1}{1-e^{-sT}}G(s)} y_d(s) \\ &= \frac{1}{1-e^{-sT}+G(s)}\left(\left(1-e^{-sT}\right)\frac{1}{1-e^{-sT}}\right) \\ &= \frac{1}{1-e^{-sT}+G(s)}. \end{aligned}$$

Then, it is only required to verify whether or not the roots of the equation $1 - e^{-sT} + G(s) = 0$ are all in the left s-plane (if the real parts of the roots are strictly less than zero, then the closed-loop system is exponentially stable (Hale & Verduyn Lunel, 1993, p. 34, Corollary 7.2.)). Therefore, the tracking problem has been reduced to a stability problem of the closed-loop system.

Actuator-Focused Viewpoint. The actuator-focussed viewpoint in the following will give a new explanation on IMP without using transfer functions. In this example, I_M is an internal model whose Laplace transform is $1/(1 - e^{-sT})$. The term

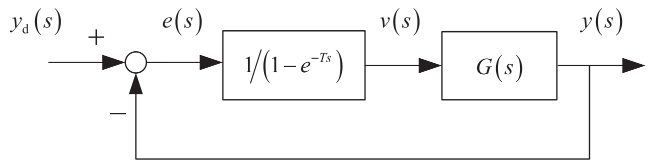

Figure 4. Periodic signal tracking of an RC system.

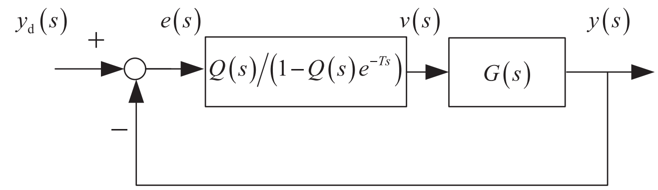

Figure 5. Periodic signal tracking of an FRC system.

$I_M(s_0, I_M(s_0', 0))$ in Theorem 3.1 can be written as

$$v(t) = v(t-T) + e(t)$$
$$e(t) = e(t-T)$$

where $v(\theta) = s_0(\theta) \in \mathbb{R}$, $e(\theta) = s_0'(\theta) \in \mathbb{R}$, $\theta \in [-T, 0]$. Given any $\theta \in [-T, 0]$, one has

$$v(kT+\theta) = s_0(\theta) + ks_0'(\theta)$$

which is not a period signal if $s_0'(\theta) \neq 0, \theta \in [-T, 0]$. Therefore, the conditions (i) of Theorem 3.1 holds, namely the composition $I_M(s_0, I_M(s_0', 0)) \notin \mathcal{I}_M$ if $s_0'(\theta) \neq 0$, $\theta \in [-T, 0]$. Obviously, the conditions (iii) of Theorem 3.1 also holds. If the closed-loop system without external signals is exponentially stable, then, according to the solution of functional functions (Hale & Verduyn Lunel, 1993), it can be proven that $v(t)$ and $e(t)$ will both tend to T-periodic signals when the system is driven by a T-periodic signal. Therefore, $v^*, e^* \in \mathcal{I}_M$. According to Theorem 3.1, it can be claimed that $e^* = 0$. An intuitive interpretation is also given by contradiction. One has

$$e(t) = v(t) - v(t-T). \tag{7}$$

Consequently, based on (7), it can be concluded that $e(t) \to 0$ as $t \to \infty$ if $v^* \in \mathcal{I}_M$ (a T-periodic signal). As shown above, to examine the tracking error tending to zero as $t \to \infty$, it only requires verifying whether or not the closed-loop system without external signals is exponentially stable. This implies that the tracking problem has been reduced to a stability problem.

Remark 3.4: Another way is given to explain why $v(t)$ and $e(t)$ will tend to T-periodic solutions as $t \to \infty$. If the closed-loop system without external signals is exponentially stable, then when the system is driven by a T-periodic signal, the closed-loop system is uniformly bounded and uniformly ultimately bounded. By the fixed point theory (or see Lemma 2.1), there exists a T-periodic solution for $v(t)$ and $e(t)$. Since the closed-loop system without external signals is exponentially stable, $v(t)$ and $e(t)$ will tend to the T-periodic solution as $t \to \infty$.

3.3 Filtered repetitive control systems subject to T-periodic signals

How to stabilise an RC system is not an easy problem due to the inclusion of the time delay element in the positive feedback loop. It was proven in Hara et al. (1988) that stability of RC systems could be achieved for continuous-time systems only when the plants are proper but not strictly proper. Moreover, the internal model $1/(1 - \mathrm{e}^{-sT})$ may lead to instability of the system. The stability of RC systems is insufficiently robust. Taking these into account, low-pass filters are introduced into RCs to enhance the stability of RC systems, resulting in FRCs which can improve the robustness of the closed-loop systems. With an appropriate filter, the FRC can usually achieve a satisfactory tradeoff between tracking performance and stability, which in turn broadens its application in practice. For example: the model $Q(s)/(1 - Q(s)\,\mathrm{e}^{-sT})$ replaces $1/(1 - \mathrm{e}^{-sT})$ resulting in the closed-loop system shown in Figure 5. Furthermore, if $Q(s) = 1/(1 + \epsilon s)$, then the relationship between $v(t)$ and $e(t)$ is

$$e(t) = v(t) - v(t-T) + \epsilon\dot{v}(t). \tag{8}$$

If the closed-loop system without external signals is exponentially stable, then, when the system is driven by a periodic signal, it is easy to see that $v(t)$ and $e(t)$ will both tend to periodic signals as $t \to \infty$. Because of the relationship (8), it can be concluded that $e(t) - \epsilon\dot{v}(t) \to 0$. This implies that the tracking error can be adjusted by the filter $Q(s)$ or say ϵ. Moreover, if $\dot{v}(t)$ is bounded in t uniformly with respect to (w.r.t) ϵ as $\epsilon \to 0$, then $\lim\limits_{t\to\infty,\epsilon\to 0} e(t, \epsilon) = 0$. On the other hand, increasing ϵ can improve the stability of the closed-loop system. Therefore, a satisfactory tradeoff between stability and tracking performance can be achieved by using the FRC.

4. Actuator-focussed RC design method

By the aforementioned actuator-focussed viewpoint, the actuator-focussed RC design method is further proposed to establish conditions that the viewpoint requires. The periodic signal tracking problem for linear periodic systems is considered first, which can be taken as a special case of nonlinear periodic systems. Then, the periodic signal tracking problem for nonlinear periodic systems is considered.

4.1 Linear periodic system

Consider the following linear periodic system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{B}(t)\,\mathbf{u}(t) + \mathbf{d}(t)\\ \mathbf{y}(t) &= \mathbf{C}^{\mathrm{T}}(t)\,\mathbf{x}(t) + \mathbf{D}(t)\,\mathbf{u}(t)\end{aligned} \tag{9}$$

where matrices $\mathbf{A}(t+T) = \mathbf{A}(t) \in \mathbb{R}^{n\times n}$, $\mathbf{B}(t+T) = \mathbf{B}(t) \in \mathbb{R}^{n\times m}$, $\mathbf{C}(t+T) = \mathbf{C}(t) \in \mathbb{R}^{n\times m}$, and $\mathbf{D}(t+T) = \mathbf{D}(t) \in \mathbb{R}^{m\times m}$ are bounded; $\mathbf{x}(t) \in \mathbb{R}^n$ is the system state, $\mathbf{u}(t) \in \mathbb{R}^m$ is the control input, $\mathbf{d} \in \mathcal{C}_T^0([0,\infty), \mathbb{R}^n)$ is a T-periodic disturbance. The objective of the control input $\mathbf{u}$ is to make $\mathbf{y}(t)$ track a T-periodic desired signal $\mathbf{y}_\mathrm{d} \in \mathcal{C}_T^0([0,\infty); \mathbb{R}^m)$.

For the system (9), similar to Equation (8), an FRC is taken in the form as

$$\mathbf{A}_{\epsilon}\dot{\mathbf{v}}(t) = -\mathbf{v}(t) + (\mathbf{I}_m - \alpha\mathbf{A}_{\epsilon})\,\mathbf{v}(t-T) + \mathbf{L}_1(t)\,\mathbf{e}(t)$$
$$\mathbf{u}(t) = \mathbf{L}_2(t)\,\mathbf{x}(t) + \mathbf{v}(t) \tag{10}$$

where $\mathbf{e} \triangleq \mathbf{y}_{\rm d} - \mathbf{y}$, $\mathbf{A}_{\epsilon} \in \mathbb{R}^{m\times m}$ is a positive semi-definite matrix, $\alpha > 0$, $\mathbf{L}_1(t+T) = \mathbf{L}_1(t) \in \mathbb{R}^{m\times m}$ is nonsingular and $\mathbf{L}_2(t+T) = \mathbf{L}_2(t) \in \mathbb{R}^{m\times n}$. Moreover, $\mathbf{L}_1(t)$ and $\mathbf{L}_2(t)$ are bounded. The introduction of variable α is used for stability analysis easier like in Quan and Cai (2011, Theorem 1) (also see the role of α in (A3) and (A4)), which is often a small positive value. Then

$$\mathbf{y}(t) = \left(\mathbf{C}^{\rm T}(t) + \mathbf{D}(t)\,\mathbf{L}_2(t)\right)\mathbf{x}(t) + \mathbf{D}(t)\,\mathbf{v}(t).$$

Next, by combining the system (9) and FRC (10), the resulting closed-loop system is written as follows

$$\mathbf{E}\dot{\mathbf{z}}(t) = \mathbf{A}_a(t)\,\mathbf{z}(t) + \mathbf{A}_d\mathbf{z}(t-T) + \mathbf{B}_a(t)\,\mathbf{w}(t) \tag{11}$$

where

$$\mathbf{z} = \begin{bmatrix}\mathbf{v}\\ \mathbf{x}\end{bmatrix},\quad \mathbf{w} = \begin{bmatrix}\mathbf{y}_{\rm d}\\ \mathbf{d}\end{bmatrix},$$
$$\mathbf{A}_a = \begin{bmatrix} -\mathbf{I}_m - \mathbf{L}_1\mathbf{D} & -\mathbf{L}_1\left(\mathbf{C}^{\rm T} + \mathbf{D}\mathbf{L}_2\right)\\ \mathbf{B} & \mathbf{A} + \mathbf{B}\mathbf{L}_2\end{bmatrix}$$
$$\mathbf{E} = \mathrm{diag}\left(\mathbf{A}_{\epsilon}, \mathbf{I}_n\right), \mathbf{A}_d = \mathrm{diag}\left(\mathbf{I}_m - \alpha\mathbf{A}_{\epsilon}, \mathbf{0}\right),$$
$$\mathbf{B}_a = \mathrm{diag}\left(\mathbf{L}_1, \mathbf{I}_n\right).$$

Lemma 4.1: *Suppose that the solution $\mathbf{z}(t) = \mathbf{0}$ of the differential equation*

$$\mathbf{E}\dot{\mathbf{z}}(t) = \mathbf{A}_a(t)\,\mathbf{z}(t) + \mathbf{A}_d\mathbf{z}(t-T) \tag{12}$$

is globally exponentially stable. Then the resulting closed-loop system in (11) *has a unique globally exponentially stable T-periodic solution $\mathbf{z}^*$.*

Proof: Since $\mathbf{z}(t) = \mathbf{0}$ of (12) is globally exponentially stable, the solutions of the resulting closed-loop system (11) are uniformly bounded and uniformly ultimately bounded. Then the resulting closed-loop system in (11) has a T-periodic solution according to Burton (1985, pp. 249–251). Suppose, to the contrary, that (11) has two solutions, denoted by $\mathbf{z}_1^*, \mathbf{z}_2^*$, satisfying

$$\mathbf{E}\dot{\mathbf{z}}_1^*(t) = \mathbf{A}_a(t)\,\mathbf{z}_1^*(t) + \mathbf{A}_d\mathbf{z}_1^*(t-T) + \mathbf{B}_a(t)\,\mathbf{w}(t) \tag{13}$$
$$\mathbf{E}\dot{\mathbf{z}}_2^*(t) = \mathbf{A}_a(t)\,\mathbf{z}_2^*(t) + \mathbf{A}_d\mathbf{z}_2^*(t-T) + \mathbf{B}_a(t)\,\mathbf{w}(t). \tag{14}$$

Subtracting (14) from (13) results in

$$\mathbf{E}\dot{\mathbf{z}}_{\rm e}^*(t) = \mathbf{A}_a(t)\,\mathbf{z}_{\rm e}^*(t) + \mathbf{A}_d\mathbf{z}_{\rm e}^*(t-T)$$

where $\mathbf{z}_{\rm e}^* = \mathbf{z}_1^* - \mathbf{z}_2^*$. Since the solution $\mathbf{z}(t) = \mathbf{0}$ of (12) is globally exponentially stable, $\mathbf{z}_{\rm e}^* = \mathbf{0}$, which implies $\mathbf{z}_1^* = \mathbf{z}_2^*$. Therefore the resulting closed-loop system in (11) has a unique stable T-periodic solution $\mathbf{z}^* = \mathbf{z}_1^* = \mathbf{z}_2^*$. ■

Theorem 4.1: *Suppose that* (12) *is globally exponentially stable. Then, the resulting closed-loop system in* (11) *has a unique globally exponentially stable T-periodic solution $\mathbf{z}^* = [\mathbf{v}^{*\rm T}\ \mathbf{x}^{*\rm T}]^{\rm T}$. Furthermore,*

$$\|\mathbf{e}\|_a \le \sup_{t\in[0,T]}\left\|\mathbf{L}_1^{-1}(t)\,\mathbf{A}_{\epsilon}\right\|\left(\|\dot{\mathbf{v}}\|_a + \alpha\,\|\mathbf{v}\|_a\right). \tag{15}$$

If $\mathbf{z}(t) = \mathbf{0}$ in (12) *is globally exponentially stable uniformly w.r.t $\mathbf{A}_{\epsilon}$ as $\|\mathbf{A}_{\epsilon}\| \to 0$, then*

$$\lim_{\|\mathbf{A}_{\epsilon}\|\to 0}\|\mathbf{e}(\mathbf{A}_{\epsilon})\|_a = 0.$$

Proof: By Lemma 4.1, the resulting closed-loop system in (11) has a unique globally exponentially stable T-periodic solution $\mathbf{z}^*$. By using (10), it follows that

$$\mathbf{L}_1(t)\,\mathbf{e}(t) = \mathbf{A}_{\epsilon}\dot{\mathbf{v}}(t) + \mathbf{v}(t) - (1 - \alpha\mathbf{A}_{\epsilon})\,\mathbf{v}(t-T).$$

Taking $\|\cdot\|_a$ on both sides of the equation above yields

$$\begin{aligned}\|\mathbf{e}\|_a &= \limsup_{t\to\infty}\left\|\begin{array}{c}\mathbf{L}_1^{-1}(t)\,\mathbf{A}_{\epsilon}\left(\dot{\mathbf{v}}(t) + \alpha\mathbf{v}(t-T)\right)\\ +\mathbf{L}_1^{-1}(t)\left(\mathbf{v}(t) - \mathbf{v}(t-T)\right)\end{array}\right\|\\ &\le \limsup_{t\to\infty}\left\|\mathbf{L}_1^{-1}(t)\,\mathbf{A}_{\epsilon}\left(\dot{\mathbf{v}}(t) + \alpha\mathbf{v}(t-T)\right)\right\|\\ &\quad + \limsup_{t\to\infty}\left\|\mathbf{L}_1^{-1}(t)\left(\mathbf{v}(t) - \mathbf{v}(t-T)\right)\right\|\\ &\le \sup_{t\in[0,T]}\left\|\mathbf{L}_1^{-1}(t)\,\mathbf{A}_{\epsilon}\right\|\left(\|\dot{\mathbf{v}}\|_a + \alpha\,\|\mathbf{v}\|_a\right)\end{aligned} \tag{16}$$

where the condition that the solutions of (11) approach the T-periodic solution is used so that

$$\limsup_{t\to\infty}\left\|\mathbf{L}_1^{-1}(t)\left(\mathbf{v}(t) - \mathbf{v}(t-T)\right)\right\| = 0.$$

If (12) is globally exponentially stable uniformly w.r.t $\mathbf{A}_{\epsilon}$ as $\|\mathbf{A}_{\epsilon}\| \to 0$, then $\|\dot{\mathbf{v}}\|_a + \alpha\|\mathbf{v}\|_a$ is bounded uniformly w.r.t $\mathbf{A}_{\epsilon}$ as $\|\mathbf{A}_{\epsilon}\| \to 0$. Consequently, $\|\mathbf{A}_{\epsilon}\|(\|\dot{\mathbf{v}}\|_a + \alpha\|\mathbf{v}\|_a) \to 0$ as $\|\mathbf{A}_{\epsilon}\| \to 0$. This implies that $\|\mathbf{e}(\mathbf{A}_{\epsilon})\|_a \to 0$ as $\|\mathbf{A}_{\epsilon}\| \to 0$ by (16). ■

A sufficient condition is given in Theorem 4.2 to ensure that $\mathbf{z}(t) = \mathbf{0}$ in (11) is globally exponentially stable.

Theorem 4.2: *If $\mathbf{A}_{\epsilon} > \mathbf{0}$ and there exist matrices $\mathbf{P} = \mathbf{P}^{\rm T} \in \mathbb{R}^{n\times n}$, $\mathbf{0} < \mathbf{Q} = \mathbf{Q}^{\rm T} \in \mathbb{R}^{m\times m}$, $\lambda_1 > 0$ such that*

$$\mathbf{0} < \mathbf{P}\mathbf{E} + \mathbf{E}^{\rm T}\mathbf{P} \tag{17}$$
$$\begin{bmatrix}\mathbf{P}\mathbf{A}_a(t) + \mathbf{A}_a^{\rm T}(t)\,\mathbf{P} + \mathbf{Q} & \mathbf{P}\mathbf{A}_d\\ \mathbf{A}_d^{\rm T}\mathbf{P} & -\mathbf{Q}\end{bmatrix} \le -\lambda_1\mathbf{I}_{n+m} \tag{18}$$

then $\mathbf{z}(t) = \mathbf{0}$ in (12) *is globally exponentially stable. In particular, if $\mathbf{A}_{\epsilon} = \mathbf{0}$,* (18) *holds and there exists $\lambda_2 > 0$ such that*

$$\sup_{t\in[0,T]}\left\|\left(\mathbf{I}_m + \mathbf{L}(t)\,\mathbf{D}(t)\right)^{-1}\right\| < 1 \tag{19}$$
$$\begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \lambda_2\mathbf{I}_{n+m}\end{bmatrix} \le \mathbf{P}\mathbf{E} + \mathbf{E}^{\rm T}\mathbf{P} \tag{20}$$

then $\mathbf{z}(t) = \mathbf{0}$ in (12) *is globally exponentially stable.*

Proof: See Appendix A.1. ■

Remark 4.1: By Theorem 4.1, the periodic signal tracking problem for linear periodic systems (9) can be converted to a stability problem (12) during which error dynamics are not required. It should be noted that the FRC (10) still works if the disturbance $\mathbf{d}$ in system (9) is unmatched. This is an advantage of the proposed actuator-focussed RC design method.

Remark 4.2: As for Theorem 4.2, it is only a sufficient condition for globally exponential stability of (12) independent of the period T, where the condition (17) is used for establishing a Lyapunov function (see Section A.1 or Quan, Yang, Cai, and Jiang (2009)). Since

$$\det(\mathbf{PE}) = \det(\mathbf{P})\det(\mathbf{E}) = \det(\mathbf{P})\det(\mathbf{A}_\epsilon)$$

one has $\det(\mathbf{PE}) = 0$ if $\det(\mathbf{A}_\epsilon) = 0$, where $\det(\cdot)$ is the determinant of a matrix. In this case, (17) does not hold. According to this, $\mathbf{A}_\epsilon > 0$ is necessary for (17). When $\mathbf{A}_\epsilon = \mathbf{0}$, the condition (19) implies $\mathbf{D} \neq 0$. This is consistent with the result for LTI systems. It was proved in Hara et al. (1988) that, for a class of general linear plants, the exponential stability of RC systems could be achieved only when the plant is proper ($\mathbf{D} \neq 0$) but not strictly proper[1].

Remark 4.3: Since $\mathbf{A}_a(t)$ is T-periodic, the linear matrix inequalities (17) and (18) cannot be solved by commonly-used tools directly. Roughly, an easy way is to select a sufficient number of sampling points in one period, namely $t_i = i\frac{T}{M}$, $i = 0, \ldots, M, M \in \mathbb{N}$, by which $\mathbf{A}_a(t_1), \ldots, \mathbf{A}_a(t_M)$ are expected to represent for the T-periodic $\mathbf{A}_a(t)$ well. Then, the time-varying linear matrix inequality (18) is replaced with M time-invariant linear matrix inequalities at the M sampling times so that commonly-used tools are applicable. A brief analysis is given in the following. Given any $t \in [0, T]$, it is supposed $t_i \leq t \leq t_{i+1}$ without loss of generality. Let

$$\mathbf{H}(t) = \begin{bmatrix} \mathbf{PA}_a(t) + \mathbf{A}_a^{\mathrm{T}}(t)\mathbf{P} + \mathbf{Q} & \mathbf{PA}_d \\ \mathbf{A}_d^{\mathrm{T}}\mathbf{P} & -\mathbf{Q} \end{bmatrix}.$$

It is assumed that $\|\dot{\mathbf{A}}_a(t)\|$ is bounded. Then $\|\dot{\mathbf{H}}(t)\| \leq \kappa < \infty$, by confining $\mathbf{P}, \mathbf{Q}$ when solving the the linear matrix inequalities in Theorem 4.2. Given $\epsilon_{\lambda_1} > 0$, if the linear matrix inequality (18) satisfies

$$\mathbf{H}(t_i) \leq -\left(\lambda_1 + \epsilon_{\lambda_1}\right)\mathbf{I}_{n+m}, \quad i = 0, \ldots, M \tag{21}$$

then

$$\begin{aligned} \mathbf{H}(t) &= \mathbf{H}(t_i) + \int_{t_i}^{t} \dot{\mathbf{H}}(s)\,\mathrm{d}s \\ &\leq \mathbf{H}(t_i) + \int_{t_i}^{t} \left\|\dot{\mathbf{H}}(s)\right\| \mathbf{I}_{n+m}\,\mathrm{d}s \\ &\leq -\left(\lambda_1 + \epsilon_{\lambda_1}\right)\mathbf{I}_{n+m} + \kappa(t - t_i)\mathbf{I}_{n+m} \\ &\leq -\left(\lambda_1 + \epsilon_{\lambda_1}\right)\mathbf{I}_{n+m} + \kappa\frac{T}{M}\mathbf{I}_{n+m}. \end{aligned}$$

If

$$\frac{T\kappa}{\epsilon_{\lambda_1}} \leq M$$

then

$$\mathbf{H}(t) \leq -\lambda_1\mathbf{I}_{n+m}. \tag{22}$$

This implies that if the sampling points are sufficient enough, then (21) implies (22).

Remark 4.4: Generally, the convergence speed mainly depends on the choice of $\mathbf{L}_1, \mathbf{L}_2$ as they are feedback gains, while the tracking error mainly depends on the choice of $\mathbf{A}_\epsilon, \alpha$ according to inequality (15).

4.2 General nonlinear system

In the following, let us consider a class of nonlinear periodic systems

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(t, \mathbf{x}, \mathbf{u}, \mathbf{d}) \\ \mathbf{y} &= \mathbf{g}(\mathbf{x}, \mathbf{u}) \end{aligned} \tag{23}$$

where $\mathbf{f} : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$, $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, and $\mathbf{f}(t, \mathbf{x}, \mathbf{u}, \mathbf{d}(t)) = \mathbf{f}(t + T, \mathbf{x}, \mathbf{u}, \mathbf{d}(t + T))$; $\mathbf{x}(t) \in \mathbb{R}^n$ is the system state, $\mathbf{u}(t) \in \mathbb{R}^m$ is the control input, $\mathbf{d} \in \mathcal{C}_T^0([0, \infty), \mathbb{R}^l)$ is the T-periodic disturbance. The objective of the control input $\mathbf{u}$ is to make $\mathbf{y}(t)$ track T-periodic desired signal $\mathbf{y}_\mathrm{d} \in \mathcal{C}_T^0([0, \infty); \mathbb{R}^m)$.

For the system (23), similar to (8), an FRC is taken in the form as

$$\begin{aligned} \mathbf{A}_\epsilon\dot{\mathbf{v}}(t) &= -\mathbf{v}(t) + (1 - \alpha\mathbf{A}_\epsilon)\mathbf{v}(t - T) + \mathbf{h}(t, \mathbf{e}) \\ \mathbf{u}(t) &= \mathbf{u}_{\mathrm{st}}(\mathbf{x}(t)) + \mathbf{v}(t) \end{aligned} \tag{24}$$

where $\mathbf{e} \triangleq \mathbf{y}_\mathrm{d} - \mathbf{y}$, $\mathbf{A}_\epsilon \in \mathbb{R}^{m\times m}$ is a positive semi-definite matrix, $\alpha > 0$, $\mathbf{h} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function, and $\mathbf{u}_{\mathrm{st}} : \mathbb{R}^n \to \mathbb{R}^m$ is a state feedback law employed to stabilise the state of the considered plant (23). The functions $\mathbf{h}(\cdot)$ and $\mathbf{u}_{\mathrm{st}}(\cdot)$ are both locally Lipschitz. On the other hand, the continuous function $\mathbf{v}$ represents a feedforward input which will drive the output $\mathbf{y}$ of (23) to track the given desired trajectory $\mathbf{y}_\mathrm{d}$. Next, the resulting closed-loop system is written as follows

$$\mathbf{E}\dot{\mathbf{z}} = \mathbf{f}_a(t, \mathbf{z}_t, \mathbf{w}) \tag{25}$$

where

$$\begin{aligned} \mathbf{z} &= [\mathbf{v}^{\mathrm{T}} \quad \mathbf{x}^{\mathrm{T}}]^{\mathrm{T}}, \quad \mathbf{w} = [\mathbf{y}_\mathrm{d}^{\mathrm{T}} \quad \mathbf{d}^{\mathrm{T}}]^{\mathrm{T}} \\ \mathbf{E} &= \mathrm{diag}(\mathbf{A}_\epsilon, \mathbf{I}_n), \quad \mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}_{\mathrm{st}}(\mathbf{x}) + \mathbf{v}) \\ \mathbf{f}_a(t, \mathbf{z}_t, \mathbf{w}) &= \begin{bmatrix} -\mathbf{v} + (1 - \alpha\mathbf{A}_\epsilon)\mathbf{v}(t - T) - \mathbf{h}(t, \mathbf{e}) \\ \mathbf{f}(t, \mathbf{x}, \mathbf{u}_{\mathrm{st}}(\mathbf{x}) + \mathbf{v}, \mathbf{d}) \end{bmatrix}. \end{aligned}$$

Theorem 4.3: *Suppose* (i) *the solutions of the resulting closed-loop system in* (25) *are uniformly bounded and uniformly ultimately bounded;* (ii) $\mathbf{h}(t, \mathbf{e}) \to \mathbf{0}$ *implies* $\mathbf{e} \to \mathbf{0}$. *Then the resulting closed-loop system in* (25) *has a T-periodic solution* $\mathbf{z}^* = [\mathbf{v}^{*\mathrm{T}}$

$\mathbf{x}^{*\mathrm{T}}]^{\mathrm{T}}$. *Let* $\mathbf{z}_{\mathrm{e}} \triangleq \mathbf{z} - \mathbf{z}^{*}$. *Furthermore, if*

$$\mathbf{E}\dot{\mathbf{z}}_{\mathrm{e}} = \mathbf{f}_a\left(t, \mathbf{z}_t^{*} + \mathbf{z}_{\mathrm{e}t}, \mathbf{w}\right) - \mathbf{f}_a\left(t, \mathbf{z}_t^{*}, \mathbf{w}\right) \tag{26}$$

is locally (globally) exponentially stable, then the T-periodic solution $\mathbf{z}^{*}$ *is locally (globally) exponentially stable and*

$$\|\mathbf{h}(\cdot, \mathbf{e})\|_a \le \|\mathbf{A}_\epsilon\| \left(\|\dot{\mathbf{v}}\|_a + \alpha \|\mathbf{v}\|_a\right)$$

holds locally (globally). Furthermore, if $\|\dot{\mathbf{v}}(\mathbf{A}_\epsilon)\|_a$ *and* $\|\mathbf{v}(\mathbf{A}_\epsilon)\|_a$ *are bounded uniformly w.r.t* $\mathbf{A}_\epsilon$ *as* $\|\mathbf{A}_\epsilon\| \to 0$, *then* $\lim_{\|\mathbf{A}_\epsilon\| \to 0} \|\mathbf{e}(\mathbf{A}_\epsilon)\|_a = 0$ *locally (globally).*

Proof: Since $\mathbf{w}$ is a T-periodic function and $\mathbf{f}(t, \mathbf{x}, \mathbf{u}, \mathbf{d}(t)) = \mathbf{f}(t + T, \mathbf{x}, \mathbf{u}, \mathbf{d}(t + T))$, one has $\mathbf{f}_a(t, \mathbf{z}_t, \mathbf{w}(t)) = \mathbf{f}_a(t + T, \mathbf{z}_t, \mathbf{w}(t + T))$. Furthermore, $\mathbf{f}_a(t, \mathbf{z}_t, \mathbf{w})$ is locally Lipschitz and the solutions of the resulting closed-loop system (25) are uniformly bounded and uniformly ultimately bounded. Then the resulting closed-loop system in (25) has a T-periodic solution according to Lemma 2.1. Since $\mathbf{z} = \mathbf{z}^{*} + \mathbf{z}_{\mathrm{e}}$, one has (26). If (26) is locally (globally) exponentially stable, $\mathbf{z}_{\mathrm{e}} \to \mathbf{0}$ as $t \to \infty$ locally (globally). This implies $\mathbf{z} \to \mathbf{z}^{*}$ as $t \to \infty$ locally (globally), namely the T-periodic solution $\mathbf{z}^{*}$ is locally (globally) exponentially stable. By using (24), it follows that

$$\mathbf{h}(t, \mathbf{e}) = \mathbf{A}_\epsilon \dot{\mathbf{v}}(t) + \mathbf{v}(t) - (1 - \alpha \mathbf{A}_\epsilon)\mathbf{v}(t - T). \tag{27}$$

Taking $\|\cdot\|_a$ on both sides of the equation (27) yields

$$\begin{aligned} \|\mathbf{h}(\cdot, \mathbf{e})\|_a &= \limsup_{t\to\infty} \left\| \begin{array}{c} \mathbf{A}_\epsilon\left(\dot{\mathbf{v}}(t) + \alpha \mathbf{v}(t - T)\right) \\ +\mathbf{v}(t) - \mathbf{v}(t - T) \end{array} \right\| \\ &\le \limsup_{t\to\infty} \|\mathbf{A}_\epsilon\left(\dot{\mathbf{v}}(t) + \alpha \mathbf{v}(t - T)\right)\| \\ &\quad + \limsup_{t\to\infty} \|\mathbf{v}(t) - \mathbf{v}(t - T)\| \\ &\le \|\mathbf{A}_\epsilon\| \left(\|\dot{\mathbf{v}}\|_a + \alpha \|\mathbf{v}\|_a\right) \end{aligned}$$

where the condition that the solutions of (25) approach the T-periodic solution is used. If $\|\dot{\mathbf{v}}\|_a$ and $\|\mathbf{v}\|_a$ are bounded uniformly w.r.t $\mathbf{A}_\epsilon$ as $\|\mathbf{A}_\epsilon\| \to 0$, then $\|\mathbf{A}_\epsilon\|(\|\dot{\mathbf{v}}\|_a + \alpha\|\mathbf{v}\|_a) \to 0$ as $\|\mathbf{A}_\epsilon\| \to 0$. This implies that $\|\mathbf{h}(\cdot, \mathbf{e}.)\|_a \to 0$ as $\|\mathbf{A}_\epsilon\| \to 0$. Note that $\mathbf{h}(t, \mathbf{e}) \to \mathbf{0}$ implies $\mathbf{e} \to \mathbf{0}$ as $t \to \infty$. Then $\lim_{\|\mathbf{A}_\epsilon\| \to 0} \|\mathbf{e}(\mathbf{A}_\epsilon)\|_a = 0$. ■

Remark 4.5: The state feedback law $\mathbf{u}_{\mathrm{st}}(\cdot)$ employed is to stabilise the state of the considered plant (23), i.e. making condition (i) of Theorem 4.3 hold. It is required that $\mathbf{h}(t, \mathbf{e}) \to \mathbf{0}$ imply $\mathbf{e} \to \mathbf{0}$ according to condition (ii) of Theorem 4.3. The major idea of the actuator-focussed RC design is to make $\mathbf{h}(t, \mathbf{e})$ as the input of the internal model, i.e. $\mathbf{A}_\epsilon \dot{\mathbf{v}}(t) = -\mathbf{v}(t) + (1 - \alpha \mathbf{A}_\epsilon)\mathbf{v}(t - T) + \mathbf{h}(t, \mathbf{e})$ appearing in (24). If the closed-loop system tends to equilibrium, then the tracking error can be analysed according to the RC itself. This is based on the actuator-focussed viewpoint.

Remark 4.6: The major advantage of the proposed actuator-focussed RC design is to avoid the derivation of error dynamics. This facilitates the tracking controller design. Through incorporating the internal model into the closed-loop system, it is only necessary to ensure that the latter is uniformly bounded and uniformly ultimately bounded. Uniform boundedness and uniformly ultimate boundedness are often related to the exponential stability of the closed-loop system when the exogenous (reference and disturbance) signals are themselves fixed identically at zero.

5. Numerical examples

In order to demonstrate its effectiveness, the actuator-focussed RC design method is further proposed to solve three periodic signal tracking problems for a linear periodic system (time-varying), a minimum-phase nonlinear system and a nonminimum-phase nonlinear system.

5.1 A linear periodic system

Consider a linear periodic system (9) with

$$\begin{aligned} \mathbf{A}(t) &= \begin{bmatrix} 0 & 1 \\ -1 - 0.3\sin t & -2 - 0.6\cos t \end{bmatrix}, \\ \mathbf{B}(t) &= \begin{bmatrix} 0.3\sin t \\ 1 \end{bmatrix}, \\ \mathbf{d}(t) &= \begin{bmatrix} 0 \\ \sin(t + 1) \end{bmatrix}, \quad \mathbf{C}(t) = \begin{bmatrix} 1 \\ 0.6\cos t \end{bmatrix}, \\ \mathbf{D}(t) &= 1. \end{aligned}$$

The objective is to design u to drive the signal $y(t) - y_{\mathrm{d}}(t) \to 0$, where $y_{\mathrm{d}}(t) = \sin t$ for simplicity. For the system above, according to FRC (10), design

$$\begin{aligned} &\epsilon \dot{v}(t) = -v(t) + (1 - 0.01\epsilon)\, v(t - T) + L_1\left(y_{\mathrm{d}}(t) - y(t)\right) \\ &u(t) = v(t), \quad v(\theta) = 0, \;\; \theta \in [-T, 0] \end{aligned} \tag{28}$$

where $L_1 = 6$. Let $\rho(t)$ be the maximal eigenvalue of the matrix at the left side of (18). If $\rho(t) < 0, \forall\, t \in [0, 2\pi]$, then (18) holds. The matrices $\mathbf{P}$ and $\mathbf{Q}$ in Theorem 4.2 can be found as

$$\mathbf{P} = \begin{bmatrix} 23.39 & 11.63 & -11.42 \\ 11.63 & 87.32 & 16.72 \\ -11.42 & 16.72 & 56.98 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} 47.11 & 24.32 & -27.13 \\ 24.31 & 27.87 & -9.08 \\ -27.13 & -9.08 & 34.18 \end{bmatrix}.$$

With them, the curve $\rho(t)$ is plotted in Figure 6 with different values $\epsilon = 0, 0.1, 1$. As shown, (18) holds. Meanwhile, with the same matrices $\mathbf{P}$ and $\mathbf{Q}$, the condition (17) holds when $\epsilon = 0.1, 1$, the conditions (19) and (20) hold when $\epsilon = 0$.

According to Theorem 4.2, the designed controller with $\epsilon = 0, 0.1, 1$ can make the closed-loop system uniformly bounded and uniformly ultimately bounded. When $\epsilon = 0$, by the actuator-focussed viewpoint, the control form (28) is to establish an input-output relation as follows

$$y_{\mathrm{d}}(t) - y(t) = \frac{1}{L_1}\left(v(t) - v(t - T)\right).$$

Since v approaches a T-periodic signal, it can be concluded that $y_{\mathrm{d}}(t) - y(t) \to 0$ as $t \to \infty$. When $\epsilon = 0.1, 1$, the tracking error $y_{\mathrm{d}}(t) - y(t)$ is uniformly ultimately bounded. With

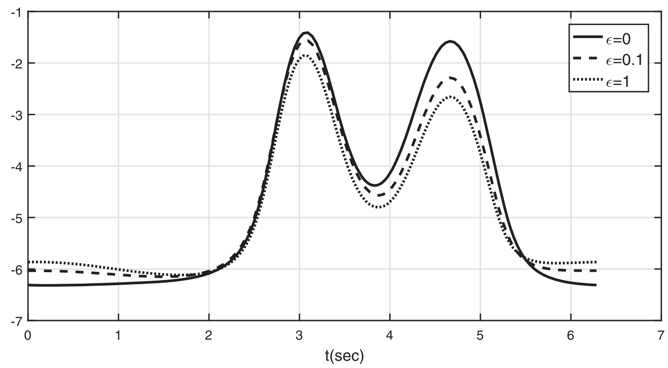

Figure 6. The curve $\rho(t)$ in one period.

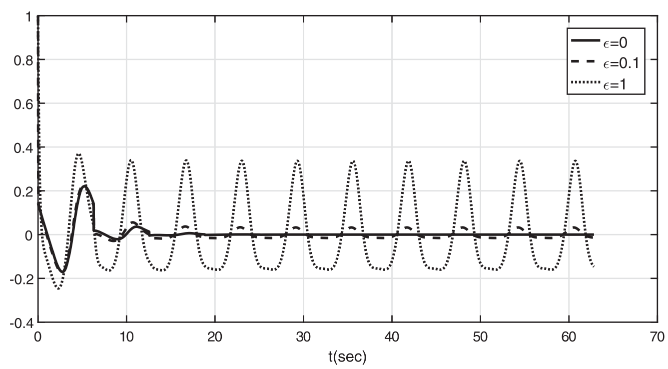

Figure 7. Tracking error for the linear periodic system with different values ϵ.

different values $\epsilon = 0, 0.1, 1$, the corresponding tracking errors are shown in Figure 7, where the tracking error is nearly zero after 20s when $\epsilon = 0$, and tracking error is greatest when $\epsilon = 1$. Therefore, the simulation is consistent with our analysis in Theorem 4.1.

5.2 A minimum-phase nonlinear system

The dynamics of an m-degree-of-freedom manipulator are described by the following differential equation

$$\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \mathbf{u} \tag{29}$$

where $\mathbf{q} \in \mathbb{R}^m$ denotes the vector of generalised displacements in robot co-ordinates, $\mathbf{u} \in \mathbb{R}^m$ denotes the vector of generalised control input forces in robot coordinates; $\mathbf{D}(\mathbf{q}) \in \mathbb{R}^{m\times m}$ is the manipulator inertial matrix, $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{m\times m}$ is the vector of centripetal and Coriolis torques and $\mathbf{G}(\mathbf{q}) \in \mathbb{R}^m$ is the vector of gravitational torques. It is assumed that both $\mathbf{q}$ and $\dot{\mathbf{q}}$ are available from measurements. Because of no internal dynamics, the system (29) is a minimum-phase nonlinear system. Two common assumptions in the following are often made on the system (29) (Lewis, Abdallah, & Dawson, 1993; Spong & Vidyasagar, 1989).

(A1) The inertial matrix $\mathbf{D}(\mathbf{q})$ is symmetric, uniformly positive definite and bounded, i.e.

$$0 < \underline{\lambda}_D \mathbf{I}_m \le \mathbf{D}(\mathbf{q}) \le \bar{\lambda}_D \mathbf{I}_m, \quad \forall\, \mathbf{q} \in \mathbb{R}^m \tag{30}$$

where $\underline{\lambda}_D, \bar{\lambda}_D > 0$.

(A2) The matrix $\dot{\mathbf{D}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric, hence

$$\mathbf{x}^{\mathrm{T}}\left(\dot{\mathbf{D}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\right)\mathbf{x} = 0, \quad \forall\, \mathbf{x} \in \mathbb{R}^m.$$

For a given desired trajectory $\mathbf{q}_{\mathrm{d}} \in \mathcal{C}^2_{PT}([0, \infty), \mathbb{R}^m)$, the controller $\mathbf{u}$ is designed to make $\mathbf{q}$ track $\mathbf{q}_{\mathrm{d}}$. Define a new state $\mathbf{x}$ as follows

$$\mathbf{x} = \dot{\mathbf{q}} + \mu\mathbf{q}$$

where $\mu > 0$. Let $0 < \mathbf{M} = \mathbf{M}^{\mathrm{T}} \in \mathbb{R}^{m\times m}$ be a positive definite matrix and $k > 0$. According to (24), a control law $\mathbf{u}$ is taken in the form as

$$\begin{aligned} \epsilon\dot{\mathbf{v}}(t) &= -\mathbf{v}(t) + (1 - \alpha\epsilon)\mathbf{v}(t - T) + k\left((\dot{\mathbf{q}}_{\mathrm{d}} + \mu\mathbf{q}_{\mathrm{d}}) - \mathbf{x}\right)(t) \\ \mathbf{u}(t) &= \mathbf{v}(t) - \mathbf{M}\mathbf{x}(t) + \mathbf{G}(\mathbf{q}(t)) \\ &\quad - \mu\mathbf{D}(\mathbf{q}(t))\dot{\mathbf{q}}(t) - \mu\mathbf{C}(\mathbf{q}(t), \dot{\mathbf{q}}(t))\mathbf{q}(t) \end{aligned} \tag{31}$$

where $\mathbf{v}(\theta) = \mathbf{0}$, $\theta \in [-T, 0]$. Substituting the controller (31) into (29) results in

$$\begin{aligned} \epsilon\dot{\mathbf{v}}(t) &= -\mathbf{v}(t) + (1 - \alpha\epsilon)\mathbf{v}(t - T) \\ &\quad + k\left((\dot{\mathbf{q}}_{\mathrm{d}} + \mu\mathbf{q}_{\mathrm{d}}) - \mathbf{x}\right)(t) \\ \dot{\mathbf{x}}(t) &= -\mathbf{D}^{-1}(\mathbf{q}(t))\left(\mathbf{C}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) + \mathbf{M}(t)\right)\mathbf{x}(t) \\ &\quad + \mathbf{D}^{-1}(\mathbf{q}(t))\mathbf{v}(t). \end{aligned} \tag{32}$$

The closed-loop system (32) can be rewritten in the form of (25). Suppose (i) Assumptions (A1)–(A2) hold, (ii) $0 < \alpha\epsilon < 1$, $\epsilon, \alpha, k > 0$. Then the solutions of the closed-loop system (32) are uniformly bounded and uniformly ultimately bounded (See Appendix A.2). Then, the solutions of closed-loop system (32) are uniformly ultimately bounded. Therefore, the closed-loop system has a T-periodic solution by Lemma 2.1. By the actuator-focussed viewpoint, according to (32), the control term $\mathbf{v}$ is to establish an input-output relationship as follows

$$\mathbf{x}_{\mathrm{d}}(t) - \mathbf{x}(t) = \frac{1}{k}\left(\dot{\mathbf{v}}(t) + \mathbf{v}(t) - (1 - \alpha\epsilon)\mathbf{v}(t - T)\right).$$

Suppose $\mathbf{q}_{\mathrm{d}} = [\sin t \;\; \cos t]^{\mathrm{T}}$ with periodicity $T = 2\pi$. The parameters of manipulator are chosen as in Khalil (2002, p. 642). The controller parameters are chosen as follows

$$\mathbf{M} = 100\mathbf{I}_2, \quad \mu = 1, \quad \alpha = 0.1, \quad \epsilon = 0.01, \quad k = 200.$$

From the simulation, $\mathbf{v}$ approaches a T-periodic solution, then

$$\mathbf{x}_{\mathrm{d}}(t) - \mathbf{x}(t) - \frac{\epsilon}{k}(\dot{\mathbf{v}} + \alpha\mathbf{v})(t) \to \mathbf{0}.$$

Therefore, it is expected that $\mathbf{q}$ can track $\mathbf{q}_{\mathrm{d}}$ with good precision. As shown in Figure 8, $\mathbf{q}$ has tracked $\mathbf{q}_{\mathrm{d}}$ with good precision in the 8th cycle. In the controller design, the stability of the closed-loop system is only considered rather than the stability of the error dynamics. As observed as above, the controller design is simple and the tracking result is satisfactory.

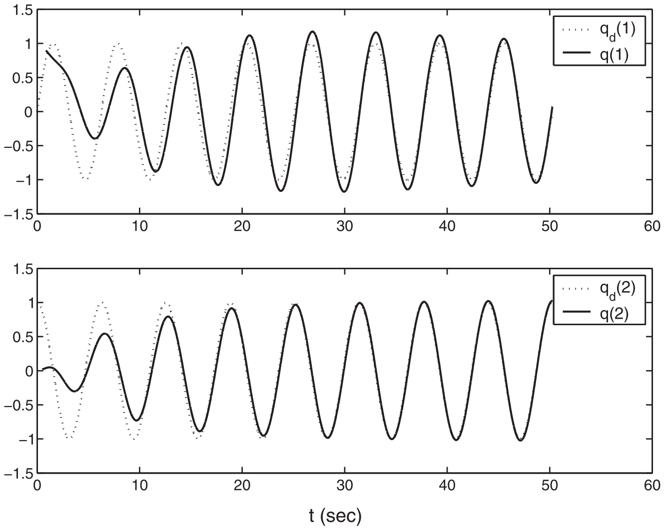

Figure 8. Two-degree-of-freedom manipulator tracking.

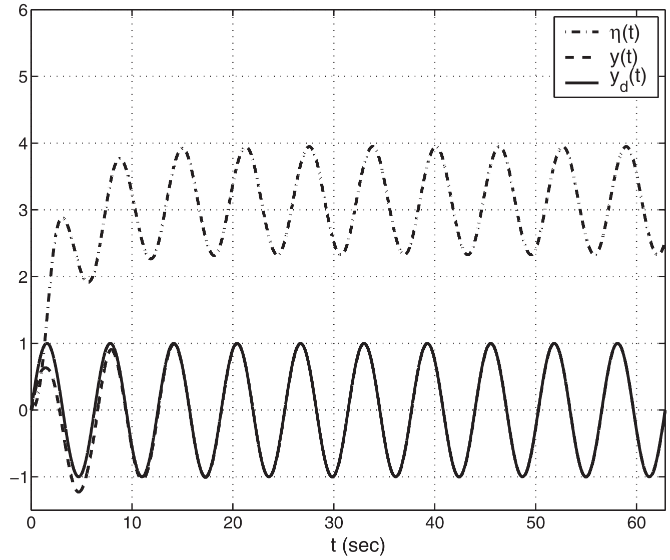

Figure 9. Periodic signal tracking of an FRC system.

Remark 5.1: For the similar problem, another way is to transform the tracking problem for nonlinear system (29) to a rejection problem for nonlinear error dynamics in the form of Kim and Ha (2000), Sun et al. (2006) and Quan and Cai (2011)

$$\dot{\mathbf{e}}_q(t) = \mathbf{f}\left(t, \mathbf{e}_q\right) + \mathbf{b}\left(t, \mathbf{e}_q\right)(\mathbf{v} - \mathbf{v}_{\mathrm{d}}). \tag{33}$$

where $\mathbf{e}_q \triangleq \mathbf{x}_{\mathrm{d}}(t) - \mathbf{x}(t)$, $\mathbf{v}_{\mathrm{d}}$ is a unknown T-periodic signal which needs to be learned by $\mathbf{v}$, the functions $\mathbf{f} : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^m$ and $\mathbf{b} : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$ are nonlinear. As for the proposed method, the closed-loop system (32) rather than error dynamics (33) is analysed, which facilitates the controller design and simplifies the designed controllers.

5.3 A nonminimum-phase nonlinear system

Consider the following nonlinear system

$$\begin{aligned}\dot{\eta} &= \sin\eta + \xi + d_\eta \\ \dot{\xi} &= u + d_\xi \\ y &= \xi\end{aligned} \tag{34}$$

where $\eta(t), \xi(t), y(t) \in \mathbb{R}$, $d_\eta, d_\xi \in C_T^0([0, \infty), \mathbb{R})$ are T-periodic disturbances. Since zero dynamics $\dot{\eta} = \sin\eta$ is unstable, the system (34) is a *nonminimum-phase* nonlinear system. The control is required not only to cause y to track y_{d}, but also to make the internal dynamics bounded. If existing methods are used to handle this problem, then it may be difficult to obtain the ideal internal dynamics because the disturbance in the internal dynamics is unknown. To the authors' knowledge, general methods handle such a case only at high computational cost (Shkolnikov & Shtessel, 2002). Compared with the existing design, the proposed design method will simplify the controller design. According to (24), a control law u is taken in the form as

$$\begin{aligned}\epsilon\dot{v}(t) &= -v(t) + (1 - \alpha\epsilon)\, v(t - T) + k\left(y_{\mathrm{d}} - y\right)(t) \\ u(t) &= -\left(q_1 + \cos\eta\right)\left(-q_1\eta + z\right)(t) - \rho z(t) \\ &\quad - q_2\eta(t) + v(t)\end{aligned} \tag{35}$$

where $v(\theta) = 0$, $\theta \in [-T, 0]$, $v, \alpha, \epsilon, k, q_1, q_2, \rho \in \mathbb{R}$ and $z = \xi + q_1\eta + \sin\eta$. Substituting the controller (35) into (34) results in

$$\begin{aligned}\epsilon\dot{v}(t) &= -v(t) + (1 - \alpha\epsilon)\, v(t - T) \\ &\quad - k\left(z - q_1\eta - \sin\eta\right)(t) + k y_{\mathrm{d}}(t) \\ \dot{\eta}(t) &= -q_1\eta(t) + z(t) + d_\eta(t) \\ \dot{z}(t) &= -kz(t) - q_2\eta(t) + v(t) + d_\xi(t) \\ &\quad + d_\eta(t)\left(q_1 + \cos\eta\right)(t).\end{aligned} \tag{36}$$

It can be proven that the solutions of the resulting closed-loop system (36) are uniformly bounded and uniformly ultimately bounded (see Appendix A.3). Then the closed-loop system has a T-periodic solution by Lemma 2.1. Suppose $d_\eta = 0.1\sin t$, $d_\xi = 0.2\sin t$, and $y_{\mathrm{d}} = \sin t$. The controller parameters are chosen as follows

$$\epsilon = 0.1, \quad \alpha = 0.1, \quad k = 5, \quad q_1 = q_2 = 1, \quad \rho = 2.$$

From the simulation, v approaches a T-periodic solution, then

$$\left(y_{\mathrm{d}} - y\right)(t) - \frac{\epsilon}{k}\left(\dot{v} + \alpha v\right)(t) \to 0.$$

Therefore, it is expected that y can track y_{d} with good precision. Figure 9 shows the response of the closed-loop system from the given initial condition. The output tracks the desired trajectory very quickly in the second cycle. The internal state $\eta(t)$ is also bounded. In the controller design, the stability of the closed-loop system is only considered rather than the stability of the error dynamics so that the derivation of the ideal internal dynamics is avoided.

Remark 5.2: Let us recall the method in Shkolnikov and Shtessel (2002), which is used to solve the similar problem. The internal dynamics of (34) is

$$\dot{\eta} = \sin\eta + \xi + d_\eta. \tag{37}$$

Since d_η is unknown, its estimate, namely $\hat{d}_\eta$, should be obtained by an observer first. Let η_d be the reference state of η. The nonlinear ideal internal dynamics is

$$\dot{\eta}_\mathrm{d} = \sin\eta_\mathrm{d} + y_\mathrm{d} + d_\eta \tag{38}$$

where ξ in (37) has been replaced by its reference y_d. According to Shkolnikov and Shtessel (2002), (38) should be linearised as

$$\dot{\eta}_\mathrm{d} = \eta_\mathrm{d} + y_\mathrm{d} + d_\eta \tag{39}$$

for computation in the following. Furthermore, by assuming that the forcing $y_\mathrm{d} + \hat{d}_\eta$ can be piecewise modelled by a linear exosystem with known characteristic polynomial, the estimate of η_d, namely $\hat{\eta}_\mathrm{d}$, is generated by a designed differential function related to (39). With $\hat{\eta}_\mathrm{d}$ obtained, the tracking problem can be converted to be a stabilising control problem. As shown, solving the observer and the differential function are time consuming, especially when the dimension of the whole system is large. Moreover, the error will be generated because of twice approximations, namely $\hat{d}_\eta$ and then $\hat{\eta}_\mathrm{d}$. Therefore, the proposed method here facilitates the controller design and simplifies the designed controllers.

6. Conclusions

A new viewpoint, namely the actuator-focussed viewpoint, on IMP is proposed in this paper. It can be used to explain how internal models work in the time domain. Compared with the cancelation viewpoint, the proposed viewpoint can be applied to nonlinear systems. Compared with the geometrical viewpoint, the proposed viewpoint is more suitable to explain how the periodic internal model, an infinite-dimensional internal model, works in the time domain. Guided by the actuator-focussed viewpoint, the actuator-focussed RC design method is further proposed for periodic signal tracking. In the controller design, the stability of the closed-loop system needs to be considered rather than that of the error dynamics. In order to demonstrate its effectiveness, the proposed design method is applied to RC problems for a linear periodic system (time-varying), a minimum-phase nonlinear system and a nonminimum-phase nonlinear system. From the given examples, the controller design is simple, and the tracking result is satisfactory. Furthermore, from the nonminimum-phase nonlinear system tracking example, the proposed design method provides a possible way to deal with some currently difficult problems.

Stochastic systems are receiving more and more attention (Wang & Zhu, 2015, 2017, 2018a, 2018b; Zhu, 2018; Zhu & Wang, 2018). The proposed actuator-focussed design method has a potential for stochastic systems, because, for which, random periodic solutions can also be obtained according to fixed point theorems (Feng, Zhao, & Zhou, 2011). Then, the relationship (7) is focussed on, where the tracking error will be determined by stochastic variables. This can be as a future work.

Note

1. In control theory, a proper transfer function is a transfer function in which the order of the numerator is not greater than the order of the denominator. A strictly proper transfer function is a transfer function where the order of the numerator is less than the order of the denominator.

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Appendix

A.1 Proof of Theorem 4.2

Choose a Lyapunov functional as

$$V(\mathbf{z}_t) = \frac{1}{2}\mathbf{z}^{\mathrm{T}}(t)\left(\mathbf{PE} + \mathbf{E}^{\mathrm{T}}\mathbf{P}\right)\mathbf{z}(t) + \int_{-T}^{0}\mathbf{z}_t^{\mathrm{T}}(\theta)\,\mathbf{Q}\mathbf{z}_t(\theta)\,\mathrm{d}\theta.$$

Taking its derivative along (12) yields

$$\begin{aligned}\dot{V}(\mathbf{z}_t) &= \mathbf{z}^{\mathrm{T}}(t)\left(\mathbf{PE} + \mathbf{E}^{\mathrm{T}}\mathbf{P}\right)\dot{\mathbf{z}}(t) + \mathbf{z}^{\mathrm{T}}(t)\,\mathbf{Qz}(t)\\ &\quad - \mathbf{z}^{\mathrm{T}}(t-T)\,\mathbf{Qz}(t-T)\\ &\le -\lambda_1\|\mathbf{z}(t)\|^2 - \lambda_1\|\mathbf{z}(t-T)\|^2 \le 0 \end{aligned} \tag{A1}$$

where (17) is utilised. Based on (A1), two conclusions are proven in the following.

(i) $\mathbf{z}(t) = \mathbf{0}$ in (12) is globally exponentially stable when $\mathbf{A}_\epsilon > 0$. For system (12), there exist $\gamma_1, \gamma_2, \rho > 0$ such that

$$\begin{aligned}\gamma_1\|\mathbf{z}(t)\|^2 \le V(\mathbf{z}_t) &\le \gamma_2\|\mathbf{z}(t)\|^2 + \rho\int_{-T}^{0}\|\mathbf{z}_t(\theta)\|^2\,\mathrm{d}\theta\\ \dot{V}(\mathbf{z}_t) &\le -\lambda_1\|\mathbf{z}(t)\|^2.\end{aligned}$$

According to Quan and Cai (2012, Theorem 1), (12) is globally exponential convergence. Futhermore, $\mathbf{z}(t) = \mathbf{0}$ is globally exponentially stable according to the stability definition.

(ii) If $\sup_{t\in[0,T]}\|(\mathbf{I}_m + \mathbf{L}_1(t)\mathbf{D}(t))^{-1}\| < 1$, then $\mathbf{z}(t) = \mathbf{0}$ in (12) is globally exponentially stable when $\mathbf{A}_\epsilon = \mathbf{0}$. For system (12), there exist $\gamma_2, \rho > 0$ such that

$$\begin{aligned}\lambda_2\|\mathbf{x}(t)\|^2 \le V(\mathbf{z}_t) &\le \gamma_2\|\mathbf{z}(t)\|^2 + \rho\int_{-T}^{0}\|\mathbf{z}_t(\theta)\|^2\,\mathrm{d}\theta\\ \dot{V}(\mathbf{z}_t) &\le -\lambda_1\|\mathbf{z}(t)\|^2.\end{aligned}$$

Similar to the proof in Quan and Cai (2012, Theorem 1), $\mathbf{x}(t)$ is globally exponential convergence. Arranging (10) results in

$$\begin{aligned}\mathbf{v}(t) &= (\mathbf{I}_m + \mathbf{L}_1(t)\,\mathbf{D}(t))^{-1}\,\mathbf{v}(t-T)\\ &\quad - (\mathbf{I}_m + \mathbf{L}_1(t)\,\mathbf{D}(t))^{-1}\,\mathbf{L}_1(t)\left(\mathbf{C}^{\mathrm{T}}(t) + \mathbf{D}(t)\,\mathbf{L}_2(t)\right)\mathbf{x}(t).\end{aligned} \tag{A2}$$

Since $\sup_{t\in[0,T]}\|(\mathbf{I}_m + \mathbf{L}(t)\mathbf{D}(t))^{-1}\| < 1$ and $\mathbf{x}(t)$ is globally exponential convergence, then $\mathbf{v}(t) = \mathbf{0}$ is globally exponential convergence as well. Consequently, according to the stability definition, $\mathbf{z} = [\mathbf{v}^{\mathrm{T}}\ \mathbf{x}^{\mathrm{T}}]^{\mathrm{T}} = \mathbf{0}$ is globally exponentially stable.

A.2 Uniformly ultimate boundedness proof for minimum-phase nonlinear system

Design a Lyapunov functional to be

$$\begin{aligned}V(\mathbf{z}_t) &= \frac{k}{2}\mathbf{x}^{\mathrm{T}}(t)\,\mathbf{D}(\mathbf{q}(t))\,\mathbf{x}(t) + \frac{\epsilon}{2}\mathbf{v}^{\mathrm{T}}(t)\,\mathbf{v}(t)\\ &\quad + \frac{1}{2}\int_{-T}^{0}\mathbf{v}_t^{\mathrm{T}}(\theta)\,\mathbf{v}_t(\theta)\,\mathrm{d}\theta.\end{aligned}$$

where (A1) is utilised. Taking the derivative of V along the solutions of (32) results in

$$\begin{aligned}\dot{V}(\mathbf{z}_t) &= -k\mathbf{x}^{\mathrm{T}}(t)\,\mathbf{M}\mathbf{x}(t) + k\mathbf{x}^{\mathrm{T}}(t)\,\mathbf{v}(t)\\ &\quad + \frac{1}{2}\left(\mathbf{v}^{\mathrm{T}}(t)\,\mathbf{v}(t) - \mathbf{v}^{\mathrm{T}}(t-T)\,\mathbf{v}(t-T)\right)\\ &\quad + \mathbf{v}^{\mathrm{T}}(t)\left(-\mathbf{v}(t) + (1-\alpha\epsilon)\,\mathbf{v}(t-T)\right)\\ &\quad + k\mathbf{v}^{\mathrm{T}}(t)\left(\mathbf{x}_{\mathrm{d}}(t) - \mathbf{x}(t)\right)\\ &\le -k\mathbf{x}^{\mathrm{T}}(t)\,\mathbf{M}\mathbf{x}(t) - \frac{\alpha\epsilon\,(2-\alpha\epsilon)}{2}\mathbf{v}^{\mathrm{T}}(t)\,\mathbf{v}(t)\\ &\quad + k\mathbf{v}^{\mathrm{T}}(t)\,\mathbf{x}_{\mathrm{d}}(t)\end{aligned}$$

where (A2) is utilised. Since

$$\varepsilon\mathbf{v}^{\mathrm{T}}\mathbf{v} + 2k\mathbf{v}^{\mathrm{T}}\mathbf{x}_{\mathrm{d}} + \frac{1}{\varepsilon}k^2\mathbf{x}_{\mathrm{d}}^{\mathrm{T}}\mathbf{x}_{\mathrm{d}} \ge 0$$

for any $\varepsilon > 0$, one has

$$\dot{V} \le -k\mathbf{x}^{\mathrm{T}}\mathbf{M}\mathbf{x} - \left(\frac{\alpha\epsilon\,(2-\alpha\epsilon)}{2} - \varepsilon\right)\mathbf{v}^{\mathrm{T}}\mathbf{v} + \frac{1}{\varepsilon}k^2\mathbf{x}_{\mathrm{d}}^{\mathrm{T}}\mathbf{x}_{\mathrm{d}}. \tag{A3}$$

Here, ε is chosen sufficiently small so that $\frac{\alpha\epsilon(2-\alpha\epsilon)}{2} - \varepsilon > 0$. Therefore, the given Lyapunov functional satisfies

$$\gamma_0\,\|\mathbf{z}(t)\|^2 \le V(\mathbf{z}_t) \le \gamma_1\,\|\mathbf{z}(t)\|^2 + \frac{1}{2}\int_{-T}^{0}\|\mathbf{z}_t(\theta)\|^2\mathrm{d}\theta$$

$$\dot{V}(\mathbf{z}_t) \le -\gamma_2\,\|\mathbf{z}(t)\|^2 + \chi\left(\sup_{t\in[0,T]}\|\mathbf{x}_{\mathrm{d}}(t)\|^2\right)$$

where $\gamma_0 = \min(\frac{k\underline{\lambda}_D}{2}, \frac{\epsilon}{2})$, $\gamma_1 = \max(\frac{k\bar{\lambda}_D}{2}, \frac{\epsilon}{2})$, $\gamma_2 = \min(k\lambda_{\min}(\mathbf{M}), \frac{\alpha\epsilon(2-\alpha\epsilon)}{2} - \varepsilon)$ and function χ belongs to class $\mathcal{K}$ (Khalil, 2002, Defintion 4.2, p. 144). According to Quan and Cai (2012, Theorem 1), the solutions of (32) are uniformly bounded and uniformly ultimately bounded.

A.3 Uniformly ultimate boundedness proof for nonminimum-phase nonlinear system

Design a Lyapunov functional to be

$$V(\mathbf{z}_t) = \frac{1}{2}p_1\eta^2(t) + \frac{1}{2}p_2z^2(t) + \frac{\epsilon}{2}v^2(t) + \frac{1}{2}\int_{-T}^{0}v_t^2(\theta)\,\mathrm{d}\theta.$$

where $\mathbf{z} \triangleq [\eta\ z\ v]^{\mathrm{T}}$ and $p_1, p_2, \epsilon > 0$. Taking the derivative of V along the solutions of (36) results in

$$\begin{aligned}\dot{V}(\mathbf{z}_t) &= p_1\eta(t)\,\dot{\eta}(t) + p_2z(t)\,\dot{z}(t) + \epsilon v(t)\,\dot{v}(t)\\ &\quad + \frac{1}{2}\left(v^2(t) - v^2(t-T)\right)\\ &= -p_1q_1\eta^2(t) - p_2kz^2(t) - \frac{\alpha\epsilon\,(2-\alpha\epsilon)}{2}v^2(t)\\ &\quad + \left(p_1 - p_2q_2\right)\eta(t)\,z(t) + \left(p_2 - \rho\right)z(t)\,v(t)\\ &\quad + v\left(q_1\eta(t) + \sin\eta(t)\right)\\ &\quad + p_1\eta(t)d_\eta(t) + p_2\left(d_\xi(t) - d_\eta(t)\left(q_1(t) + \cos\eta(t)\right)\right)z(t)\\ &\quad + \rho v(t)y_{\mathrm{d}}(t).\end{aligned}$$

By fixing p_1, q_1 and choosing $p_2 = \rho$ and $0 < \alpha\epsilon < 1$, if k is chosen sufficiently large, then

$$\begin{aligned}&-p_1q_1\eta^2 - p_2kz^2 - \frac{\alpha\epsilon\,(2-\alpha\epsilon)}{2}v^2 + \left(p_1 - p_2q_2\right)\eta z\\ &\quad + \left(p_2 - \rho\right)zv + v\left(q_1\eta + \sin\eta\right) \le -\theta_1\eta^2 - \theta_2z^2 - \theta_3v^2\end{aligned} \tag{A4}$$

where $\theta_1, \theta_2, \theta_3 > 0$. Furthermore, there exists a class $\mathcal{K}$ function $\chi : [0,\infty) \to [0,\infty)$ (Khalil, 2002, Defintion 4.2, p. 144) such that

$$\begin{aligned}\dot{V} &\le -\theta_1'\eta^2 - \theta_2'z^2 - \theta_3'v^2\\ &\quad + \chi\left(\sup_{t\in[0,T]}\left(\left\|y_{\mathrm{d}}(t)\right\|^2 + \left\|d_\eta(t)\right\|^2 + \left\|d_\xi(t)\right\|^2\right)\right)\end{aligned}$$

where $\theta_1', \theta_2', \theta_3' > 0$. Therefore, the given Lyapunov functional satisfies

$$\gamma_0\,\|\mathbf{z}(t)\|^2 \le V(\mathbf{z}_t) \le \gamma_1\,\|\mathbf{z}(t)\|^2 + \frac{1}{2}\int_{-T}^{0}\|\mathbf{z}_t(\theta)\|^2\mathrm{d}\theta$$

$$\begin{aligned}\dot{V}(\mathbf{z}_t) &\le -\gamma_2\,\|\mathbf{z}(t)\|^2\\ &\quad + \chi\left(\sup_{t\in[0,T]}\left(\left\|y_{\mathrm{d}}(t)\right\|^2 + \left\|d_\eta(t)\right\|^2 + \left\|d_\xi(t)\right\|^2\right)\right)\end{aligned}$$

where $\gamma_0 = \min(\frac{1}{2}p_1, \frac{1}{2}p_2, \frac{\epsilon}{2})$, $\gamma_1 = \max(\frac{1}{2}p_1, \frac{1}{2}p_2, \frac{\epsilon}{2})$ and $\gamma_2 = \min(\theta_1', \theta_2', \theta_3')$. According to Quan and Cai (2012, Theorem 1), the solutions of (36) are uniformly bounded and uniformly ultimately bounded. Furthermore, ξ is also uniformly ultimately bounded by using the relationship $\xi = z - q_1\eta - \sin\eta$.