Repetitive Control for Nonlinear Systems
Lesson 02 Preliminaries

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What are the common-used and necessary preliminaries in the designing a repetitive controller?
Outline

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   - Sensitivity and Complementary Sensitivity Function
   - Small Gain Theorem
   - Positive Real Systems

2. **Transformation Related Preliminaries**
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   - Input-Output Linearization
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3. **Stability Related Preliminaries**
   - Barbalat’s Lemma
   - Ordinary Differential Equation
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4. **Rejection Problem and Tracking Problem**

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Noncausal Zero Phase Filter

- For a *causal system*, its response does not begin before the input function is applied, i.e., its response depends on present and past inputs only and not on future inputs. For example, for $s$-transformation, causal systems are in the form as
  \[ Q_1(s) = e^{-0.08s}, \quad Q_2(s) = \frac{1}{s}. \]
  For $z$-transformation, causal systems can be in the form as
  \[ Q_1(z) = \frac{1}{z}, \quad Q_2(z) = \frac{1}{z^2 + z + 1}. \]

- A *noncausal system* does not satisfy the property of causal systems, i.e., its response depends on future values of the input functions. For example, for $s$-transformation, causal systems can be in the form as
  \[ Q_1(s) = e^{0.08s}, \quad Q_2(s) = s. \]
  For $z$-transformation, causal systems can be in the form as
  \[ Q_1(z) = z, \quad Q_2(z) = z^2 + z + 1. \]
Noncausal Zero Phase Filter

Definition 2.1

A filter is said to be zero phase when its frequency response $H(j\omega)$ for $s$-transformation (or $H(e^{j\omega})$ for $z$-transformation) is a real and even function of radian frequency $\omega \in \mathbb{R}$, and when $H(j\omega) > 0$ for $s$-transformation (or $H(e^{j\omega}) > 0$ for $z$-transformation) in the filter passband(s).

- It should be known that zero phase filters must be noncausal.
- For example

$$H_1(s) = \sum_{k=-N}^{N} a_k e^{-sk}$$

where $a_k = a_{-k}$, $a_k \in \mathbb{R}$, $k = 1, 2, \cdots N$. In Fourier space, substituting $s = j\omega$ into $H_1(s)$ results in

$$H_1(j\omega) = \sum_{k=-N}^{N} a_k e^{-j\omega k} = a_0 + 2\sum_{k=1}^{N} a_k \cos(\omega k).$$
As shown in Fig.1, the tracking error $E$ is

$$E(s) = S(s)(R(s) - D(s)) + T(s)M(s)$$  \hspace{1cm} (1)

where

$$S(s) = \frac{1}{1 + P(s)L(s)}, \quad T(s) = \frac{P(s)L(s)}{1 + P(s)L(s)}.$$

(2)
Sensitivity and Complementary Sensitivity Function

The functions $S(s)$ and $T(s)$ are the closed-loop sensitivity and complementary sensitivity transfer functions, respectively. Obviously,

$$S(s) + T(s) = 1.$$  \hspace{1cm} (3)

- As for disturbance rejection and tracking, according to Eq. (1), it is expected that $\sup_{\omega \in \mathbb{R}} |S(j\omega)|$ is as small as possible so that the term $\sup_{\omega \in \mathbb{R}} |S(j\omega)(R(j\omega) - D(j\omega))|$ is also small.
- On the other hand, it is expected that $\sup_{\omega \in \mathbb{R}} |T(j\omega)|$ is as small as possible so that the small term $\sup_{\omega \in \mathbb{R}} |T(j\omega)M(j\omega)|$ implies a good sensor noise attenuation.
- The two objectives are contradiction because of relationship (3). So the constraint on $S(s)$ exists.
- Suppose if the sensor noise is ignored. Then the tracking error becomes

$$E(s) = S(s)(R(s) - D(s)).$$  \hspace{1cm} (4)

In this case, can the closed-loop sensitivity transfer function $S(s)$ be designed freely? The answer is No.
This is revealed by *Bode's sensitivity integral* discovered by Hendrik Wade Bode

\[
\int_0^\infty \ln |S(j\omega)| \, d\omega = \int_0^\infty \ln \left| \frac{1}{1 + P(j\omega)L(j\omega)} \right| \, d\omega
\]

\[
= \pi \sum \text{Re} (p_k) - \frac{\pi}{2} \lim_{s \to \infty} sP(s)L(s)
\]

(5)

where \( p_k \in \mathbb{C} \) are the poles of \( P(s)L(s) \) in the right half plane (unstable poles), namely the zeros of \( S(s) \) in the right half plane (unstable zeros).

If \( P(s)L(s) \) has at least two more poles than zeros, Eq. (5) is simplified to

\[
\int_0^\infty \ln |S(j\omega)| \, d\omega = \pi \sum \text{Re} (p_k).
\]

Furthermore, if \( P(s)L(s) \) has no poles in the right half plane, then

\[
\int_0^\infty \ln |S(j\omega)| \, d\omega = 0.
\]
Figure: Bode’s sensitivity integral
Small Gain Theorem

The set of signals for which this norm is finite is known as the finite-horizon Lebesgue 2-space:

\[ \mathcal{L}_2[0, T] = \left\{ \|f\|_{2,[0,T]} < \infty \right\} . \]

In order to address stability issues, the behavior of signals over infinite time intervals must be considered. The infinite-horizon Lebesgue 2-space is defined by

\[ \mathcal{L}_2(-\infty, \infty) = \{ \|f\|_2 < \infty \} \]

where

\[ \|f\|_2 \triangleq \left\{ \int_{-\infty}^{\infty} \|f(t)\|^2 \, dt \right\}^{1/2} . \]

The spaces \( \mathcal{L}_2[0, \infty) \) and \( \mathcal{L}_2(-\infty, 0) \) are defined by \( \mathcal{L}_2[0, \infty) = S_+ \cap \mathcal{L}_2(-\infty, \infty) \) and \( \mathcal{L}_2(-\infty, 0) = S_- \cap \mathcal{L}_2(-\infty, \infty) \). It is convenient to introduce the extended 2-space \( \mathcal{L}_{2e} \) defined by

\[ \mathcal{L}_{2e} = \{ f \in \mathcal{L}_2[0, T], \text{ for all } T < \infty \} . \]
Small Gain Theorem

For any complex matrix $\mathbf{Q} \in \mathbb{C}^{m \times p}$, there exist $\mathbf{Y} \in \mathbb{C}^{m \times m}$, $\mathbf{U} \in \mathbb{C}^{p \times p}$ and a real matrix $\mathbf{\Sigma} \in \mathbb{R}^{r \times r}$ such that

$$
\mathbf{Q} = \mathbf{Y} \begin{bmatrix} \mathbf{\Sigma} & \mathbf{0}_{r \times (p-r)} \\ \mathbf{0}_{(m-r) \times r} & \mathbf{0}_{(m-r) \times (p-r)} \end{bmatrix} \mathbf{U}^* \tag{6}
$$

in which $\mathbf{\Sigma} = \text{diag}(\sigma_1, \cdots, \sigma_r)$ with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and $\min(m, p) \geq r$. Expression (6) is called singular value decomposition (SVD) of $\mathbf{Q}$. The maximum singular value and the minimum singular of matrix $\mathbf{Q}$ will be denoted by

$$
\sigma_{\text{max}}(\mathbf{Q}) = \sigma_1, \sigma_{\text{min}}(\mathbf{Q}) = \sigma_r.
$$

Equivalently, they can also be defined as

$$
\sigma_{\text{max}}(\mathbf{Q}) = \max_{\|\mathbf{u}\|=1} \|\mathbf{Q}\mathbf{u}\|,
$$

$$
\sigma_{\text{min}}(\mathbf{Q}) = \min_{\|\mathbf{u}\|=1} \|\mathbf{Q}\mathbf{u}\|.
$$
The feedback system given in Fig. 3 is called *internally stable* if each of the four transfer functions mapping $w_1$ and $w_2$ to $e_1$ and $e_2$ are stable.

**Figure:** Feedback loop of two systems
Small Gain Theorem

Figure: Feedback loop of two systems

Theorem 2.1

Suppose the systems $G_1: \mathcal{L}_{2e} \rightarrow \mathcal{L}_{2e}$ and $G_2: \mathcal{L}_{2e} \rightarrow \mathcal{L}_{2e}$ in Fig.3 have finite incremental gains such that $\sup_\omega \|G_1(j\omega)\| \sup_\omega \|G_2(j\omega)\| < 1$, where $\|G_i(j\omega)\| = \sigma_{\text{max}}(G_i(j\omega))$, $i = 1, 2$. Then:

1. For all $w_1, w_2 \in \mathcal{L}_{2e}$, there exist unique solutions $e_1, e_2 \in \mathcal{L}_{2e}$.

2. For all $w_1, w_2 \in \mathcal{L}_2 [0, \infty)$, there exist unique solutions $e_1, e_2 \in \mathcal{L}_2 [0, \infty)$. That is, the closed loop is internally stable.
Definition 2.3
A real symmetric matrix \( \mathbf{A} \in \mathbb{R}^{n \times n} \) is symmetric semi-positive definite if \( \mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0 \) for any nonzero \( \mathbf{x} \in \mathbb{R}^n \). Furthermore, the matrix is called a positive definite matrix if \( \mathbf{x}^T \mathbf{A} \mathbf{x} > 0 \) for any nonzero \( \mathbf{x} \in \mathbb{R}^n \).

In the following, the notion \( \mathbf{A} \geq \mathbf{0} \) or \( \mathbf{A} > \mathbf{0} \) represent that \( \mathbf{A} \) is a positive semidefinite or positive definite matrix, respectively.

Definition 2.4
The rational transfer matrix \( \mathbf{H}(s) \) is positive real if: (1) \( \mathbf{H}(s) \) has no pole in \( \text{Re}(s) > 0 \), (2) \( \mathbf{H}(s) \) is real for all positive real \( s \), (3) \( \mathbf{H}(s) + \mathbf{H}^*(s) > 0 \) for all \( \text{Re}(s) > 0 \).

Definition 2.5
A rational transfer function \( \mathbf{H}(s) \) that is not identically zero for all \( s \), is strictly positive real if \( \mathbf{H}(s - \epsilon) \) is positive real for some \( \epsilon > 0 \).
Theorem 2.2

Let \( H(s) \) be a proper rational transfer matrix\(^a\), and suppose that 
\[
\det (H(s) + H^T (-s)) \neq 0.
\]
Then \( H(s) \) is strictly positive real if and only if

1. \( H(s) \) has all its poles with negative real parts and one of the following three conditions is satisfied:
   1. \( H(\infty) + H^T (\infty) > 0 \),
   2. \( H(\infty) + H^T (\infty) = 0 \) and \( \lim_{\omega \to \infty} \omega^2 (H(j\omega) + H^T (-j\omega)) > 0 \),
   3. \( H(\infty) + H^T (\infty) \geq 0 \) (but not zero nor nonsingular) and there exist positive constants \( \sigma \) and \( \delta \) such that
      \[
      \omega^2 \sigma_{\min} (H(j\omega) + H^T (-j\omega)) \geq \sigma, \quad \forall |\omega| \geq \delta.
      \]

\(^a\)In control theory, a proper transfer function is a transfer function in which the order of the numerator is not greater than the order of the denominator.
Theorem 2.3

Consider that the system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= C^T x(t) + Du(t)
\end{align*}
\]  

is controllable and observable, where \( A \in \mathbb{R}^{n \times n}, B, C \in \mathbb{R}^{n \times m}, \)
\( D \in \mathbb{R}^{m \times m}, x \in \mathbb{R}^n, y, u \in \mathbb{R}^m. \) The transfer function
\( H(s) = C^T (sI_n - A)^T B + D \) is positive real if and only if there exists matrices \( 0 < P = P^T \in \mathbb{R}^{n \times n}, L \in \mathbb{R}^{n \times m} \) and \( W \in \mathbb{R}^{m \times m} \) such that

\[
\begin{align*}
PA + A^T P &= -LL^T \\
PB - C &= -LW \\
D + D^T &= W^T W.
\end{align*}
\]
**Theorem 2.4**

Consider the system in (7). Assume that the rational transfer matrix
\[ H(s) = C^T (sI_n - A)^T B + D \]
has poles which lie in \( \text{Re}(s) < -\gamma \), where \( \gamma > 0 \) and \( (A, B, C, D) \) is a minimal realization of \( H(s) \). Then \( H(s - \epsilon) \) for \( \epsilon > 0 \) is positive real if and only if there exist \( 0 < P = P^T \in \mathbb{R}^{n \times n}, L \in \mathbb{R}^{n \times m} \) and \( W \in \mathbb{R}^{m \times m} \) such that

\[
\begin{align*}
PA + A^TP &= -LL^T - 2\epsilon P \\
PB - C^T &= -LW \\
D + D^T &= W^TW.
\end{align*}
\]
Transformation Related Preliminaries
Schur Complement

Let $X$ be a symmetric matrix given by

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

Define $X / A$ to be the Schur complement of $A$ in $X$, namely

$$X / A = C - B^T A^{-1} B$$

and $X / C$ be the Schur complement of $C$ in $X$, namely

$$X / C = A - B^T C^{-1} B.$$ 

Then

$$X > 0 \iff A > 0, X / A > 0,$$

$$X > 0 \iff C > 0, X / C > 0.$$ 

Furthermore, if $A > 0$, then $X \geq 0 \iff X / A \geq 0$; if $C > 0$, then $X \geq 0 \iff X / C \geq 0$. 
(1) SISO Case
Take a class of single-input single-output (SISO) systems

\[ \dot{x} = f(x) + g(x) u \]
\[ y = h(x) \]  \hfill (8)

Consider a smooth scalar function \( h(x) \), whose gradient is defined as

\[ \nabla h = \frac{\partial h}{\partial x}. \]  \hfill (9)

The gradient \( \nabla h \in \mathbb{R}^{1 \times n} \) is a row vector and the \( j \)th component is denoted by \( (\nabla h)_j = \frac{\partial h}{\partial x_j} \).
Definition 2.6

Consider a smooth scalar function \( h(x) : \mathbb{R}^n \rightarrow \mathbb{R} \) and a vector field \( f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n \). The map \( x \mapsto \nabla h(x) \cdot f(x) \) is called the **Lie derivative** of the function \( h \) with respect to the vector field \( f \), and is denoted by \( L_f h \). And \( L_f h \) is thought of as the directional derivative of the function \( h \) along the integral curves of \( f \).

Furthermore, the higher-order Lie derivative is deduced as

\[
L_f^0 h = h
\]

\[
L_f^i h = L_f \left( L_f^{i-1} h \right) = \nabla \left( L_f^{i-1} h \right) f
\]

where \( L_f^i h \) denotes the \( i \)th order derivative of \( L_f h \). If \( g \) is another vector field, then the scalar function is written as

\[
L_g L_f h = \nabla \left( L_f h \right) g.
\]
With these definitions, the derivative $y$ is given by

$$\dot{y} = \nabla h \dot{x}$$

$$= L_f h + L_g hu.$$  \hspace{1cm} (10)

If $L_g h \neq 0$ in the set $x \in \Omega$, then

$$u = (L_g h)^{-1} (-L_f h + v)$$

can make Eq. (10) be

$$\dot{y} = v$$  \hspace{1cm} (11)

where $v \in \mathbb{R}$ is the new defined control input.
If $L_g h \equiv 0$ in the set $x \in \Omega$, then Eq. (10) becomes

$$\dot{y} = L_f h.$$ 

In this case, the second derivative of $y$ is given by

$$\ddot{y} = L_f^2 h + L_g L_f h u.$$ 

If $L_g h = 0$, $L_g L_f h = 0$, $\cdots$, $L_g L_f^{\rho-2} h = 0$ and $L_g L_f^{\rho-1} h \neq 0$ in the set $\Omega$, the system (8) is called to have relative degree $\rho$, where $\rho \in \mathbb{Z}_+$ and $\rho \leq n$. 
Then

\[ y^{(\rho)} = L_f^\rho h + L_g L_f^{\rho-1} h u \]

and

\[ u = \left( L_g L_f^{\rho-1} h \right)^{-1} (-L_f^\rho h + \nu) \]

can further make (10) be

\[ y^{(\rho)} = \nu. \tag{12} \]

However, \( x \) is \( n \)-dimension, whereas the input-output system (12) only has \( \rho \leq n \) dimension. So, it should be known \textit{where the disappeared state is}. This problem will be discussed in the following.
(2) Problem Formulation
Consider a multiple-input and multiple-output (MIMO) nonlinear system of the form

\[
\dot{x} = f(x) + g(x)u \\
y = h(x).
\]

The objective of the *input-output linearization* is to find a feedback transformation

\[
v = \alpha(x) + \beta(x)u
\]

and a state transformation

\[
z = \begin{bmatrix} z_i \\ z_o \end{bmatrix} = \psi(x)
\]
such that

\[
\begin{bmatrix}
\dot{z}_i \\
\dot{z}_o
\end{bmatrix} = \begin{bmatrix}
f_i(z_i, z_o) \\
A_oz_o + B_ov
\end{bmatrix}
\]

\[y = C_o^T z_o.\]  

(14)

The system (14) has two parts, namely the \textit{internal subsystem}

\[\dot{z}_i = f_i(z_i, z_o)\]

and the \textit{external subsystem}

\[\dot{z}_o = A_o z_o + B_o v.\]

Since the external part consists of a linear relation between \(y\) and \(v\), it is easy to design controllers, such as for tracking problems.
The result (12) is a special case of transformed system (14), where

\[
v = L_f^\rho h + L_g L_f^{\rho-1} hu
\]

\[
z_\circ = \begin{bmatrix} y & \dot{y} & \cdots & y^{(\rho-1)} \end{bmatrix}^T \in \mathbb{R}^\rho
\]

\[
A_\circ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \ddots & 1 \\ 0 & \cdots & 0 \end{bmatrix},
B_\circ = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},
C_\circ = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
\]
Input-Output Linearization

(3) Zero Dynamics

\[
\begin{bmatrix}
\dot{z}_i \\
\dot{z}_o
\end{bmatrix} =
\begin{bmatrix}
f_i(z_i, z_o) \\
A_\circ z_o + B_\circ v
\end{bmatrix}
\]

\[y = C_\circ^T z_o.\]

If the internal subsystem is unstable, then it may cause \(\|z_i(t)\| \to \infty\) as \(t \to \infty\), even if the output has tracked the reference. Consequently, \(\|x(t)\| \to \infty\) and then \(\|u(t)\| \to \infty\) as \(t \to \infty\). That will be meaningless to design such a controller.
In order to explain this, zero dynamics of the system (13) is defined as

\[ \dot{z}_i = f_i(z_i, 0). \]

If zero dynamics is stable, then the nonlinear system (13) is said to be minimum-phase. Otherwise, the nonlinear system (13) is said to be nonminimum-phase. Sometimes, there is no internal subsystem. For example, \( \rho = n \) in the SISO case. Such a system is minimum phase. Generally speaking, for a tracking problem, the internal subsystem is in fact required to be input-state stable for

\[ \dot{z}_i = f_i(z_i, \delta) \]

namely state \( z_i \) is bound subject to a bound input \( \delta \), where \( \delta = z_o \) for the tracking problem. In this case, such a system is minimum-phase.
For SISO LTI systems in the form of transfer functions, if a transfer function has stable zeros, then it is *minimum-phase*; otherwise, it is *nonminimum-phase*; if no zeros, it is *minimum-phase*. For example, the transfer functions 

\[ G_1(s) = \frac{s + 1}{s(s + 2)}, \quad G_2(s) = \frac{s - 1}{s(s + 2)}, \quad G_3(s) = \frac{1}{s(s + 2)} \]

are minimum phase, nonminimum phase and minimum phase, respectively.
The objective of the *input-state linearization* is to find a nonunique diffeomorphism $\psi : \Omega_x \rightarrow \mathbb{R}^n$ that

$$z = \psi(x)$$

to transform the system (13) to be

$$\dot{z} = Az + B\beta^{-1}(x)(u - \alpha(x))$$  \hspace{1cm} (15)

where $z \in \mathbb{R}^n$, $\beta(x)$ is nonsingular for $\forall x \in \Omega_x$ and $(A, B)$ completely controllable. Then, design a feedback controller

$$v = \alpha(x) + \beta(x)u$$

such that

$$\dot{z} = Az + Bv.$$  \hspace{1cm} (16)

A problem arises that the output will become $y = h(\psi^{-1}(z))$, which is still nonlinear.
Example 2.1. Consider the system as follows:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix}
= \begin{bmatrix}
x_2 \\
x_4 \\
0
\end{bmatrix}
B \left( x_1 x_4^2 - G \sin x_3 \right) + \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} u
\]

where \( x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \). The output \( y \) is successively differentiated three times until the input \( u \) appears algebraically for the first time in

\[
y^{(3)} = B x_2 x_4^2 - B G x_4 \cos x_3 + 2 B x_1 x_4 u.
\]

Note, however, that the coefficient of \( u \) is zero whenever \( x_1 \) or \( x_4 \) is zero.
Approximation Linearization

In this case, the *approximation linearization* will be applied. A nonlinear change of coordinates $z = \psi(x)$ is constructed to transform the system as follows

\[
\begin{align*}
    z_1 &= y = \psi_1(x) \\
    \dot{z}_1 &= \begin{cases} 
        x_2 \\
        z_2 = \psi_2(x)
    \end{cases} \\
    \dot{z}_2 &= -BG \sin x_3 + Bx_1x_4^2 \\
    \dot{z}_3 &= -BGx_4 \cos x_3 \\
    \dot{z}_4 &= BGx_4^2 \sin x_3 + BG \cos x_3 u.
\end{align*}
\]

So the relative degree has its definition in this set $x_3 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
Approximation Linearization

Design

\[ u = \frac{1}{BG \cos x_3} (-BGx_4^2 \sin x_3 + v) \, . \]

As a result, the equations are rearranged as

\[ \dot{z} = Az + Bv + \phi(x) \]
\[ y = C^T z \]

where

\[
\begin{bmatrix}
    z_1 \\
    z_2 \\
    z_3 \\
    z_4
\end{bmatrix},
\begin{bmatrix}
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
    0 \\
    0 \\
    0 \\
    1
\end{bmatrix},
\begin{bmatrix}
    0 \\
    0 \, Bx_1 x_4^2 \\
    0 \\
    0
\end{bmatrix},
\begin{bmatrix}
    1 \\
    0 \\
    0 \\
    0
\end{bmatrix} \, .
(1) Basic Concept Consider an ‘original’ system as follows:

\[ \dot{x} = f(t, x, u), \quad x(0) = x_0 \]  \hspace{1cm} (17)

where \( x \in \mathbb{R}^n \). First, a ‘primary’ system is brought in, having the same dimension as the original system:

\[ \dot{x}_p = f_p(t, x_p, u_p), \quad x_p(0) = x_{p,0} \]  \hspace{1cm} (18)

where \( x_p \in \mathbb{R}^n \). From the original system and the primary system, the following ‘secondary’ system is derived:

\[ \dot{x} - \dot{x}_p = f(t, x, u) - f_p(t, x_p, u_p), \quad x(0) = x_0. \]

New variables \( x_s \in \mathbb{R}^n \) are defined as follows:

\[ x_s \triangleq x - x_p, \quad u_s \triangleq u - u_p. \]  \hspace{1cm} (19)
(1) Basic Concept

Then, the secondary system can be further written as follows:

\[ \dot{x}_s = f(t, x_p + x_s, u_p + u_s) - f_p(t, x_p, u_p) \]

\[ x_s(0) = x_0 - x_{p,0}. \]  

(20)

From the definition (19), it follows

\[ x(t) = x_p(t) + x_s(t), t \geq 0. \]  

(21)

The process is shown in Fig. 5.

**Figure:** Additive State Decomposition on a Dynamical Control System Results in: A Primary system and Secondary System.
Examples Example 2.2. In fact, the idea of the additive state decomposition has been implicitly mentioned in existing literature. An existing example is the tracking controller design, which often requires a reference system to derive error dynamics. The reference system (primary system) is assumed to be given as follows:

\[
\dot{x}_r = f(t, x_r, u_r), \quad x_r(0) = x_{r,0}.
\]

Based on the reference system, the error dynamics (secondary system) are derived as follows:

\[
\dot{x}_e = f(t, x_e + x_r, u) - f(t, x_r, u_r), \quad x_e(0) = x_0 - x_{r,0}
\]

where \(x_e \triangleq x - x_r\). This is a commonly-used step to transform a tracking problem to a stabilization problem when adaptive control is used.
**Example 2.3.** Consider a class of systems as follows:

\[
\begin{aligned}
    \dot{x}(t) &= (A + \Delta A(t))x(t) + A_d x(t - T) + Br(t) \\
    e(t) &= - (C + \Delta C(t))^T x(t) + r(t) \\
    x(\theta) &= \varphi(\theta), \theta \in [-T, 0]
\end{aligned}
\]  \quad (22)

Choose (22) as the original system and design the primary system as follows:

\[
\begin{aligned}
    \dot{x}_p(t) &= Ax_p + A_d x_p(t - T) + Br(t) \\
    e_p(t) &= -C^T x_p + r(t) \\
    x_p(\theta) &= \varphi(\theta), \theta \in [-T, 0]
\end{aligned}
\]  \quad (23)
Then the secondary system is determined by the rule (20):

\[
\begin{align*}
\dot{x}_s(t) &= (A + \Delta A(t))x_s(t) + A_dx_s(t - T) + \Delta A(t)x_p(t) \\
e_s(t) &= -(C + \Delta C(t))^T x_s(t) - \Delta C^T(t)x_p(t) \\
x_s(\theta) &= 0, \quad \theta \in [-T, 0]
\end{align*}
\]

(24)

By additive state decomposition, \(e(t) = e_p(t) + e_s(t)\). Since \(\|e(t)\| \leq \|e_p(t)\| + \|e_s(t)\|\), the tracking error \(e(t)\) can be analyzed by \(e_p(t)\) and \(e_s(t)\) separately. If \(e_p(t)\) and \(e_s(t)\) are bounded and small, then so is \(e(t)\). Fortunately, note that (23) is a linear time-invariant system and is independent of the secondary system (24), for the analysis of which many tools such as the transfer function are available. By contrast, the transfer function tool cannot be directly applied to the original system (22) as it is time-varying.
Example 2.4. Consider a class of nonlinear systems as follows:

\[
\begin{align*}
\dot{x} &= Ax + Bu + \phi(y) + d, \quad x(0) = x_0 \\
y &= C^T x
\end{align*}
\]

where \(x, y, u\) represent the state, output and input, respectively; the function \(\phi(\cdot)\) is nonlinear. The objective is to design \(u\) such that \(y(t) - r(t) \to 0\) as \(t \to \infty\). Choose (25) as the original system and design the primary system as follows:

\[
\begin{align*}
\dot{x}_p &= A x_p + B u_p + \phi(r) + d, \quad x_p(0) = x_0 \\
y_p &= C^T x_p.
\end{align*}
\]
Then the secondary system is determined by the rule (20):
\[
\dot{x}_s = Ax_s + Bu_s + \phi \left( y_p + C^T x_s \right) - \phi \left( r \right), \quad x_s (0) = 0
\]
\[
y_s = C^T x_s
\]
(27)
where \( u_s \triangleq u - u_p \). Then \( x = x_p + x_s \) and \( y = y_p + y_s \). The process is shown in Fig.6.
Additive State Decomposition

(3) Comparison with Superposition Principle
A well-known example implicitly using additive state decomposition is the Superposition Principle, widely used in physics and engineering.

Superposition Principle

For all linear systems, the net response at a given place and time caused by two or more stimuli is the sum of the responses which would have been caused by each stimulus individually.

For a simple linear system

\[ \dot{x} = Ax + B(u_1 + u_2), \ x(0) = 0 \]

the statement of the superposition principle means \( x = x_p + x_s \), where

\[ \dot{x}_p = Ax_p + Bu_1, \ x_p(0) = 0 \]
\[ \dot{x}_s = Ax_s + Bu_2, \ x_s(0) = 0. \]
Obviously, this result can also be derived from the additive state decomposition. Moreover, the superposition principle and additive state decomposition have the following relationship. From Table 2.1, additive state decomposition can be applied not only to linear systems but also nonlinear systems.

Table 2.1. The relationship

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Stability Related Preliminaries
Barbalat’s Lemma

**Definition 2.7**

Suppose $g : [0, \infty) \to \mathbb{R}$. The function $g(t)$ is said to be uniformly continuous on $[0, \infty)$ if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|d| < \delta$ implies $|g(t + d) - g(t)| < \varepsilon$ for all $t$ on $[0, \infty)$. Since $g(t + d) - g(t) = \dot{g}(t + sd)d$, $s \in [0, 1]$ by the mean value theorem, the function $g(t)$ is uniformly continuous on $[0, \infty)$ if $\dot{g}$ is bounded on $[0, \infty)$.

**Lemma 2.1, Barbalat’s Lemma**

Suppose $f : [0, \infty) \to \mathbb{R}$. If the differentiable function $f(t)$ has a finite limit as $t \to \infty$, and if $\dot{f}(t)$ is uniformly continuous, then $\dot{f}(t) \to 0$ as $t \to \infty$.

A function that $f(t) \to \text{constant as } t \to \infty$ does not implies that $\dot{f}(t) \to 0$ as $t \to \infty$. A necessary condition is required that $\dot{f}(t)$ be uniformly continuous. For example, the function $f(t) = \frac{1}{t} \sin(t^2)$ will vanish, but $\dot{f}(t) = -\frac{1}{t^2} \sin(t^2) + 2 \cos(t^2) \not\to 0$. 
Ordinary Differential Equation

For simplicity, the dynamical system in the form of

$$\dot{x} = f(t, x)$$  

(28)

is further discussed, $t \geq t_0 \geq 0$, where $f(t, 0) = 0$, and $x = 0$ is a solution.

**Definition 2.8**

- For system (28), the solution $x = 0$ is said to be *stable* if, for any $t_0 \geq 0$, $\varepsilon \in \mathbb{R}_+$, there exists $\delta = \delta(\varepsilon, t_0) \in \mathbb{R}_+$, such that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \varepsilon$ for all $t \geq t_0$. Otherwise, the solution is *unstable*.

- The solution $x = 0$ is said to be *uniformly stable*, if for any $\varepsilon \in \mathbb{R}_+$, there exists $\delta = \delta(\varepsilon) \in \mathbb{R}_+$ independent of $t_0$ such that if $\|x(t_0)\| < \delta$, then $\|x(t)\| < \varepsilon$ for all $t \geq t_0$.

- For system (28), the solution $x = 0$ is *asymptotically stable* if it is stable, and there exists some $\delta = \delta(t_0) \in \mathbb{R}_+$ such that $\|x(t_0)\| < \delta$, which implies that $\|x(t)\| \to 0$ as $t \to \infty$. 
Definition 2.8

The solution \( x = 0 \) is *uniformly asymptotically stable* if it is uniformly stable, and there exists some \( \delta \in \mathbb{R}_+ \) independent of \( t_0 \) such that, for any \( \varepsilon \in \mathbb{R}_+ \), there exists a \( T = T(\varepsilon) > 0 \) satisfying

\[
\|x(t)\| < \varepsilon, \quad \forall t \geq t_0 + T(\varepsilon), \forall \|x(t_0)\| < \delta.
\]

For system (28), an equilibrium state \( x = 0 \) is *exponentially stable* if there exist \( \alpha, \lambda, \delta \in \mathbb{R}_+ \) such that

\[
\|x(t)\| \leq \alpha \|x(t_0)\| e^{-\lambda(t-t_0)}
\]

for \( \forall \|x(t_0)\| < \delta \) and \( t \geq t_0 \).
Definition 2.9

A solution \( x(t) \) to Eq. (28) with \( x(t_0) \) is said to be \textit{uniformly bounded}, if, for each \( \delta > 0 \), there exists \( \epsilon > 0 \) such that \( \|x(t)\| \leq \epsilon \), \( t \geq t_0 \), when \( \|x(t_0)\| < \delta \). The solution \( x(t) \) to Eq. (30) with \( x(t_0) \) is said to be \textit{uniformly ultimately bounded} with \textit{ultimate bound} \( \epsilon \), if for each \( \delta > 0 \) there exists \( T = T(\epsilon, \delta) > 0 \) independent of \( t_0 \) such that \( \|x(t)\| \leq \epsilon \) for all \( t \geq t_0 + T \) when \( \|x(t_0)\| < \delta \).

Definition 2.10, 2.11

- A continuous function \( \alpha : [0, a) \to [0, \infty) \) is said to belong to class \( \mathcal{K} \) if it is strictly increasing and \( \alpha(0) = 0 \). It is said to belong to class \( \mathcal{K}_\infty \) if \( a = \infty \) and \( \alpha(r) \to \infty \) as \( r \to \infty \).

- A continuous function \( \beta : [0, a) \times [0, \infty) \to [0, \infty) \) is said to belong to class \( \mathcal{KL} \) if, for each fixed \( s \), the mapping \( \beta(r, s) \) belongs to class \( \mathcal{K} \) with respect to \( r \) and, for each fixed \( r \), the mapping \( \beta(r, s) \) is decreasing with respect to \( s \) and \( \beta(r, s) \to 0 \) as \( s \to \infty \).
With these definitions, the concept of input-to-state stability (ISS) is introduced. Consider the system

$$\dot{x} = f(t, x, u)$$ (29)

where \( f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n. \)

**Definition 2.12**

The system (29) is said to be the *input-to-state stable* if there exist a class \( \mathcal{KL} \) function \( \beta \) and a class \( \mathcal{K} \) function \( \gamma \) such that for any initial state \( x(t_0) \) and any bounded input \( u(t) \), the solution \( x(t) \) exists for all \( t \geq t_0 \) and satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right).$$
The Lyapunov-like theorem that follows gives a sufficient condition for the input-to-state stability.

**Theorem 2.5**

Let $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

\[
\alpha_1 (\|x\|) \leq V(t, x) \leq \alpha_2 (\|x\|) \\
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, u) \leq -W(x), \forall \|x\| \geq \rho (\|u\|) > 0
\]

$\forall (t, x, u) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$, where $\alpha_1, \alpha_2$ are class $\mathcal{K}_\infty$ functions, $\rho$ is a class $\mathcal{K}$ function, and $W(x)$ is a continuous positive definition function on $\mathbb{R}^n$. Then, the system (29) is input-to-state stable with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$. 
Consider a general perturbed time-delay equation

$$\dot{x}(t) = f(t, x_t, w)$$  \hspace{1cm} (30)

with $$x(s) = \phi(s), s \in [-\tau, 0], \tau \in \mathbb{R}_+$$, where $$x(t) \in \mathbb{R}^n$$, $$w(t) \in \mathbb{R}^m$$ is a piecewise continuous and bounded perturbation. The function $$f : \mathbb{R}_+ \cup \{0\} \times C([-\tau, 0], \mathbb{R}^n) \times \mathbb{R}^m \rightarrow \mathbb{R}^n$$ is supposed to ensure the existence and uniqueness of the solution through every initial condition $$\phi$$.

**Definition 2.14**

- The solutions $$x_t(\phi)$$ to Eq. (30) with $$x(t_0 + s) = \phi(s), s \in [-\tau, 0]$$ are said to be **uniformly bounded**, if for each $$\delta > 0$$ there exists $$\varepsilon > 0$$ such that $$\|x(\phi)(t)\| \leq \varepsilon$$, $$t \geq t_0$$, when $$\sup_{s \in [-\tau, 0]} \|\phi(s)\| < \delta$$.

- The solutions $$x_t(\phi)$$ to Eq. (30) with $$x(t_0 + s) = \phi(s), s \in [-\tau, 0]$$ are said to be **uniformly ultimately bounded** with ultimate bound $$\varepsilon$$, if for each $$\delta > 0$$ there exists $$T = T(\varepsilon, \delta) > 0$$ independent of $$t_0$$ such that $$\|x(\phi)(t)\| \leq \varepsilon$$ for all $$t \geq t_0 + T$$ when $$\sup_{s \in [-\tau, 0]} \|\phi(s)\| < \delta$$. 
Corollary 2.1

Suppose that there exists a Lyapunov functional $V(t, x_t) : \mathbb{R}_+ \cup \{0\} \times C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}_+ \cup \{0\}$ such that

$$\gamma_1 \|x(t)\|^2 \leq V(t, x_t) \leq \gamma_2 \|x(t)\|^2 + \int_{t-T}^{t} \|x(s)\|^2 ds$$  \quad (31)

where $\gamma_1, \gamma_2$ are positive real numbers. If there exists $c \in \mathbb{R}_+$ such that

$$\dot{V}(t, x_t) \leq -\gamma_3 \|x(t)\|^2 + \sigma$$  \quad (32)

where $\gamma_3$ is a positive real number, then the solutions to Eq. (30) are uniformly ultimately bounded with respect to the bound $\sqrt{\frac{\sigma}{\gamma_1 \gamma_3} (\gamma_2 + T)}$. 
Rejection Problem and Tracking Problem
Rejection Problem and Tracking Problem

Assume the state-space representation of system $P$ in Fig. 7 to be

$$\dot{x} = f(x) + g(x)u, \quad x(0) = x_0$$
$$y = h(x)$$

(33)

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$, $u \in \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^m$ is continuous with $h(0) = 0$, and $x_0 \in \mathbb{R}^n$. The objective is to design the controller $C$ to make $e(t) \to 0$ as $t \to \infty$ ($e$ is shown in Fig. 7), while the state $x$ is bounded. As shown in Fig. 7, depending on how $r \in \mathbb{R}^m$ appears in the closed-loop, the problem above can be classified into the rejection problem and the tracking problem.
Assumption 2.1

There exists a differentiable function $V : \mathbb{R}^n \to [0, \infty)$ such that

$$V(x) \geq c_1 \|x\|^2 \quad (34)$$

$$\left( \frac{\partial V(x)}{\partial x} \right)^T f(x) \leq -c_2 \|x\|^2 \quad (35)$$

where $c_1, c_2 \in \mathbb{R}_+$. 

Assumption 2.2

The signal $r$ is constant, i.e., $\dot{r} = 0$ or $r$ is generated by

$$\dot{x}_r = 0, \quad r = h(x_r) \quad (36)$$

where $x_r \in \mathbb{R}^n$ is the desired state.
(1) **Rejection Problem.** In Fig.7(a), according to (33), the state-space representation of system $P$ is written as

$$
\dot{x} = f(x) + g(x)(v + r) \\
y = h(x)
$$

(37)

where $v \in \mathbb{R}^m$ and $u = v + r$. The control problem is formulated to design the controller $v$ to make $e(t) = -y(t) \to 0$ as $t \to \infty$, while the state $x$ is bounded. This is a *rejection problem*. 

**Figure:** Tracking problem and rejection problem.
Rewrite (37) as

\[ \dot{x} = f(x) + g(x)u \]
\[ y = h(x). \]

In order to find the controller, design a Lyapunov function as

\[ W(x) = V(x) + \frac{1}{2}u^T u. \]

Then

\[ \dot{W}(x) = \left( \frac{\partial V(x)}{\partial x} \right)^T f(x) + \left( \frac{\partial V(x)}{\partial x} \right)^T g(x)u + u^T \dot{v} \]

(38)

where \( \dot{r} = 0 \) is utilized (Assumption 2.2). According to (38), if the controller \( v \) is designed as follows

\[ \dot{v} = -g^T(x) \frac{\partial V(x)}{\partial x} \]

then, by using (34) and (35) in Assumption 2.1, it can be shown that

\[ x(t) \to 0 \text{ as } t \to \infty \text{ in (33), which further implies } y(t) \to 0 \text{ as } t \to \infty. \]
(2) Tracking Problem. In Fig. 7(b), the control problem is formulated to design the controller \( v \) to make \( e(t) = r(t) - y(t) \to 0 \) as \( t \to \infty \) in (33), while the state \( x \) is bounded. This is a tracking problem. By subtracting (36) from (33), the tracking problem is often solved based on an error system as follows

\[
\dot{z} = \tilde{f}(z, x_r) + g(x) v \\
e = \tilde{h}(z, x_r)
\] (39)

where \( z \triangleq x - x_r, \tilde{h}(z, x_r) \triangleq h(x_r) - h(z + x_r), \) and \( \tilde{f}(z, x_r) \triangleq f(z + x_r). \) From these definitions, one has \( \tilde{h}(0, x_r) \equiv 0 \) and \( \tilde{f}(0, x_r) \equiv 0. \)

- Obviously, the function \( \tilde{f} \) differs from the function \( f. \) This implies that the function \( V \) in Assumption 2.1 cannot be used directly. A new feedback control term may be needed to stabilize the error system.
- Moreover, \( x_r \) or \( z \) cannot be obtained directly.
- From the analysis above, it can be seen that the tracking problem is more difficult than the rejection problem.
Choose a topic, such as "Noncausal Zero Phase Filter", and make a 1-2 page slide to supplement.
All course source can be downloaded at http://rfly.buaa.edu.cn/publications.html. For More, please refer to the book:

Thank you!