How is the repetitive controller designed to tackle tracking problem for a class of nonlinear systems under the additive-state-decomposition-based tracking control framework?
Outline

1. Problem Formulation

2. Additive-State-Decomposition-Based RC Framework
   - Decomposition
   - Controller Design

3. Summary

4. Exercise
Problem Formulation
Consider a class of uncertain nonlinear systems

\[
\begin{align*}
\dot{x} &= Ax + Bu + \phi(y, \dot{y}) + d, \quad x(0) = x_0 \\
y &= C^T x
\end{align*}
\]  

(1)

where \( A \in \mathbb{R}^{n \times n} \) is a known stable constant matrix, \( B, C \in \mathbb{R}^{n \times m} \) are known constant matrices, \( \phi : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) is a known nonlinear function vector, \( x(t) \in \mathbb{R}^n \) is the state vector, \( y(t) \in \mathbb{R}^m \) is the output, \( u(t) \in \mathbb{R}^m \) is the control, and \( d(t) \in \mathbb{R}^n \) is an unknown bounded \( T \)-periodic disturbance. It is assumed that only \( y(t), \dot{y}(t) \in \mathbb{R}^m \) are available from measurement. The reference \( r(t) \in \mathbb{R}^m \) is a known and sufficiently smooth \( T \)-periodic signal, \( t \geq 0 \). In the following, for convenience, the variable \( t \) will be omitted except when necessary.
Problem Formulation

For system (1), the following assumption is made.

Assumption 7.1

The pair \((A, C)\) is observable.

Objective

The objective here is to design a controller \(u\) such that \(y(t) - r(t) \to 0_{m \times 1}\) as \(t \to \infty\) or with good tracking accuracy, i.e., \(y - r\) is ultimately bounded by a small value.

If the considered nonlinear system (1) is a single-input, single-output (SISO) nonminimum-phase system, i.e., the transfer function of the linear part (regardless of nonlinear dynamics \(\phi(\cdot, \cdot)\))

\[
\mathbf{C}^\text{T} (s\mathbf{I}_n - A)^{-1} \mathbf{B} = \frac{N(s)}{D(s)}
\]

is nonminimum-phase here, where \(N(s)\) has zeros on the right \(s\)-plane.
Consider a class of uncertain nonlinear systems

\[ \dot{x} = Ax + Bu + \phi(y, \dot{y}) + d, \quad x(0) = x_0 \]
\[ y = C^T x \]

Since the pair \((A, C)\) is observable under Assumption 6.5, there always exists a vector \(L \in \mathbb{R}^{n \times m}\) such that \(A + LC^T\) is stable, whose eigenvalues can be assigned freely. As a result, (1) can be rewritten as

\[ \dot{x} = (A + LC^T)x + Bu + (\phi(y, \dot{y}) - Ly) + d. \]

Therefore, we assume \(A\) to be stable without loss of generality. It is noticed that the property of nonminimum-phase cannot be changed by output feedback.
Example 7.1. Consider a simple nonminimum-phase nonlinear system as follows

\[ \begin{align*}
\dot{x}_1 &= 5x_1 - 10x_2 + \sin x_2 + d_1 \\
\dot{x}_2 &= 4x_1 - 8x_2 + u \\
y &= x_2
\end{align*} \tag{2} \]

where \( u \in \mathbb{R} \) is the control signal and \( d_1 \in \mathbb{R} \) is an unknown \( T \)-periodic signal.

- The output \( y \) is measurable.
- The objective is to design controller \( u \) such that the output \( y(t) - r(t) \to 0 \) as \( t \to \infty \), meanwhile keeping the other states bounded, where \( r(t) \) is a \( T \)-periodic reference, and its derivative is bounded.
- Since the internal dynamics of (2) are \( \dot{x}_1 = 5x_1 \), which are unstable, the considered problem is an RC problem for a nonminimum-phase nonlinear system.
The system (2) can be rewritten as

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
5 & -10 \\
4 & -8
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u +
\begin{bmatrix}
\sin x_2 \\
0
\end{bmatrix} +
\begin{bmatrix}
d_1 \\
0
\end{bmatrix}
\]

where \( A_0 \) is unstable. Since the pair \((A_0, C)\) is observable, the system (3) is further written as (1) with

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
5 & -12 \\
4 & -9
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
\sin y + 2y \\
y
\end{bmatrix}
\]

where the eigenvalues of \( A \) are \(-1, -3\). So, \( A \) is stable.
Consider the system (1) as the original system. According to the principle mentioned above, the primary system is chosen as follows

\[
\dot{x}_p = Ax_p + Bu_p + \phi(r, \dot{r}) + d
\]
\[
y_p = C^T x_p, \quad x_p(0) = x_0. \tag{4}
\]

Then the secondary system is determined by the original system (1) and the primary system (4) that

\[
\dot{x}_s = Ax_s + Bu_s + \phi \left( C^T x_p + C^T x_s, C^T \dot{x}_p + C^T \dot{x}_s \right) - \phi(r, \dot{r})
\]
\[
y_s = C^T x_s, \quad x_s(0) = 0_{n \times 1} \tag{5}
\]

where \(u_s = u - u_p\).
Then, one has
\[ x = x_p + x_s \quad \text{and} \quad y = y_p + y_s. \] (6)

The secondary system (5) is further written as
\[
\dot{x}_s = Ax_s + Bu_s + \phi \left( r + C^T x_s - e_p, \dot{r} + C^T \dot{x}_s - \dot{e}_p \right) - \phi (r, \dot{r}) \\
y_s = C^T x_s, \quad x_s (0) = 0_{n \times 1} \] (7)

where \( e_p \triangleq r - y_p \). If \( e_p \equiv 0_{m \times 1} \), then \((x_s, u_s) = 0\) is an equilibrium point of (7).
Controller design for the decomposed systems (4)-(5) will use their outputs or states as feedback. However, they are unknown.

**Theorem 7.1**

Suppose that an observer is designed to estimate $y_p$ and $x_s$ in (4)-(5) as follows

\[
\hat{y}_p = y - C^T \hat{x}_s \\
\dot{\hat{x}}_s = A\hat{x}_s + Bu_s + \phi(y, \dot{y}) - \phi(r, \dot{r}), \hat{x}_s(0) = 0_{n \times 1}.
\]

Then $\hat{y}_p \equiv y_p$ and $\hat{x}_s \equiv x_s$.

**Proof.** Subtracting (9) from (5) results in $\dot{\tilde{x}}_s = A\tilde{x}_s, \tilde{x}_s(0) = 0_{n \times 1}$, where $\tilde{x}_s = x_s - \hat{x}_s$. Then $\tilde{x}_s \equiv 0_{n \times 1}$. This implies that $\hat{x}_s \equiv x_s$. Consequently, by (6), $\hat{y}_p \equiv y - C^T \hat{x}_s$. □

\[\]
Example 7.2 (Example 7.1 Continued). According to (4), the primary system is

\[
\begin{bmatrix}
\dot{x}_{1,p} \\
\dot{x}_{2,p}
\end{bmatrix} = \begin{bmatrix} 5 & -12 \\ 4 & -9 \end{bmatrix} \begin{bmatrix} x_{1,p} \\
x_{2,p} \end{bmatrix}
\]

\[+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_p + \begin{bmatrix} \sin r + 2r + d_1 \\ r \end{bmatrix}\]

\[y_p = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,p} \\
x_{2,p} \end{bmatrix}, \begin{bmatrix} x_{1,p} (0) \\
x_{2,p} (0) \end{bmatrix} = x_0 \quad (10)\]

which is an LTI system. The objective of (10) is to drive \(y_p(t) - r(t) \to 0\) as \(t \to \infty\).
Then the secondary system is determined by the original system (2) and the primary system (10) that

\[
\begin{bmatrix}
\dot{x}_{1,s} \\
\dot{x}_{2,s}
\end{bmatrix} = \begin{bmatrix}
5 & -12 \\
4 & -9
\end{bmatrix} \begin{bmatrix}
x_{1,s} \\
x_{2,s}
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u_s \\
+ \begin{bmatrix}
sin y + 2y \\
y
\end{bmatrix} - \begin{bmatrix}
sin r + 2r \\
r
\end{bmatrix}
\]

\[y_s = \begin{bmatrix}
0 & 1
\end{bmatrix} \begin{bmatrix}
x_{1,s} \\
x_{2,s}
\end{bmatrix}, \quad \begin{bmatrix}
x_{1,s}(0) \\
x_{2,s}(0)
\end{bmatrix} = \mathbf{0}_{2\times1}
\]  

(11)

where \( y = y_p + y_s \).
The system (11) can be rewritten as

\[
\begin{align*}
\dot{x}_{1,s} &= 5x_{1,s} - 10x_{2,s} + \sin (x_{2,s} + r) - \sin r + g_{a,1} \\
\dot{x}_{2,s} &= 4x_{1,s} - 8x_{2,s} + u_s + g_{a,2} \\
y_s &= x_{2,s}
\end{align*}
\]  

(12)

where

\[g_{a,1} = 2(y_p - r) + \sin (x_{2,s} + y_p) - \sin (x_{2,s} + r)\]

and

\[g_{a,2} = y_p - r.\]

From the definitions of \(g_{a,1}\) and \(g_{a,2}\), it is clear that \(g_{a,1} \rightarrow 0\) and \(g_{a,2} \rightarrow 0\) as \(y_p - r \rightarrow 0\).
According to *Theorem 7.1*, the observer is designed to estimate $y_p$ and $x_s$ in (10)-(11) as follows:

\[
\hat{y}_p = y - C^T x_s
\]

\[
\begin{bmatrix}
\dot{\hat{x}}_{1,s} \\
\dot{\hat{x}}_{2,s}
\end{bmatrix} =
\begin{bmatrix}
5 & -12 \\
4 & -9
\end{bmatrix}
\begin{bmatrix}
\hat{x}_{1,s} \\
\hat{x}_{2,s}
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} u_s \\
+ \begin{bmatrix}
\sin y + 2y \\
y
\end{bmatrix} - \begin{bmatrix}
\sin r + 2r \\
r
\end{bmatrix}
\]

\[
\hat{x}_s (0) = 0_{2 \times 1}.
\]

Then $\hat{y}_p \equiv y_p$ and $\hat{x}_s \equiv x_s$. 

(13)
So far, the considered system has been decomposed into two systems in charge of corresponding tasks. In this section, the controller design in the form of problems with respect to the two-component tasks is investigated. The whole process is shown in Fig. 7.1.

**Problem 7.1:** Output Feedback Tracking for LTI System (4): \( y_p - r \rightarrow 0 \)

**Problem 7.2:** State Feedback Stabilization Subtask for Nonlinear System (5): \( y_s \rightarrow 0 \)

**Figure:** Additive state decomposition flow
\[
\dot{x}_p = Ax_p + Bu_p + \phi(r, \dot{r}) + d \\
y_p = C^T x_p, \quad x_p(0) = x_0.
\]

**Problem 7.1.** For (4), design an RC

\[
\begin{align*}
\dot{z}_p &= \alpha_p \left( z_p, y_p, r, \cdots, r^{(N)} \right) \\
u_p &= u_p \left( z_p, y_p, r, \cdots, r^{(N)} \right) 
\end{align*}
\]

such that \(e_p(t) \to B(0_{m \times 1}, \delta)^2\) as \(t \to \infty\), where \(\delta = \delta(r, d) > 0\) depends on the reference \(r\) and disturbance \(d\), and \(r^{(k)}\) denotes the \(k\)th derivative of \(r\), \(k = 1, \cdots, N\).

\[2B(\delta) \triangleq \{\xi \in \mathbb{R}^n | \|\xi\| \leq \delta\}, \; \delta > 0; \text{ the notation } x(t) \to B(\delta) \text{ means } \min_{y \in B(\delta)} |x(t) - y| \to 0.\]
Problem 7.2. For (7) (or (5)), design a controller

$$\dot{z}_s = \alpha_s \left( z_s, x_s, r, \cdots, r^{(N)} \right)$$

$$u_s = u_s \left( z_s, x_s, r, \cdots, r^{(N)} \right)$$  \hspace{1cm} (15)

such that (i) the closed-loop system is input-to-state stable with respect to the input $e_p$, namely

$$\|x_s(t)\| \leq \beta (\|x_s(0)\|, t) + \gamma \left( \sup_{0 \leq s \leq t} \|e_p(s)\| \right), \quad t \geq 0,$$  \hspace{1cm} (16)

where $r^{(k)}$ denotes the $k$th derivative of $r$, $k = 1, \cdots, N$, function $\beta$ is a class $KL$ function and $\gamma$ is a class $K$ function. Or (ii) the closed-loop system is asymptotical stable, namely $e_p(t) \to 0_{m \times 1}$ as $t \to \infty$. 
Controller Design

Theorem 7.2

Under Assumption 7.1, suppose (1) Problems 7.1-7.2 are solved; (2) the controller for system (1) is designed as Observer:

\[
\dot{\hat{x}}_s = A\hat{x}_s + Bu_s + \phi(y, \dot{y}) - \phi(r, \dot{r}), \hat{x}_s(0) = 0_{n \times 1}
\]

\[
\hat{y}_p = y - C^T\hat{x}_s
\]

Controller:

\[
\dot{z}_p = \alpha_p \left( z_p, \hat{y}_p, r, \ldots, r^{(N)} \right), z_p(0) = 0, \dot{z}_s = \alpha_s \left( z_s, \hat{x}_s, r, \ldots, r^{(N)} \right), z_s(0) = 0
\]

\[
u_p = u_p \left( z_p, \hat{y}_p, r, \ldots, r^{(N)} \right), u_s = u_s \left( z_p, \hat{x}_s, r, \ldots, r^{(N)} \right)
\]

\[
u = u_p + u_s.
\]

Then the output of system (1) satisfies \( y - r \rightarrow B(0_{m \times 1}, \delta + \|C\| \gamma(\delta)) \) as \( t \rightarrow \infty \). In particular, if \( \delta = 0 \), then the output in system (1) satisfies \( y(t) - r(t) \rightarrow 0_{m \times 1} \) as \( t \rightarrow \infty \).
Example 7.3 (Example 7.2 Continued). The primary system (10) is written as

\[ y_p(s) = G_p(s) u_p(s) + d_p(s) \]

where \( G_p(s) = C^T (sI_2 - A)^{-1} B = \frac{s-5}{s^2+4s+3} \) and

\[ d_p(s) = C^T (sI_2 - A)^{-1} (d_l(s) + x_0) . \]

For Problem 7.1, the RC is designed as

\[ u_p(s) = \frac{Q(s)e^{-Ts}}{1 - Q(s)e^{-Ts}} B(s) e_p(s) \]

where

\[ B(s) = \frac{400 (s^2 + 4s + 3) (-s - 5)}{25 (s + 10) (s + 5) (s + 8)} , \quad Q(s) = \frac{1}{0.1s + 1} . \]

The tracking performance of the primary system (10) driven by the controller (18) is shown in Fig 7.2.
Controller Design

As shown, the output $y_p$ tracks $r$ with good tracking accuracy, meanwhile keeping all states bounded.

Figure: The tracking performance of the primary system of (10)
Controller Design

For the secondary system

\[
\begin{align*}
\dot{x}_{1,s} &= 5x_{1,s} - 10x_{2,s} + \sin(x_{2,s} + r) - \sin r + g_{a,1} \\
\dot{x}_{2,s} &= 4x_{1,s} - 8x_{2,s} + u_s + g_{a,2} \\
y_s &= x_{2,s}
\end{align*}
\]

define

\[
z_s = 6x_{1,s} - 10x_{2,s} + \sin(x_{2,s} + r) - \sin r.
\]

Then the secondary system (12) can be written as

\[
\begin{align*}
\dot{x}_{1,s} &= -x_{1,s} + z_s + g_{a,1} \\
\dot{z}_s &= (\cos(x_{2,s} + r) - 10)(4x_{1,s} - 8x_{2,s} + u_s + g_{a,2}) \\
&\quad + 6(-x_{1,s} + z_s) + \cos(x_{2,s} + r) \dot{r} - \dot{r} \cos r.
\end{align*}
\]
Design the controller as

\[ u_s = -4x_{1,s} + 8x_{2,s} + \frac{1}{\cos(x_{2,s} + r) - 10} \left( -2 (-x_{1,s} + z_s) - \cos(x_{2,s} + r) \dot{r} + \dot{r} \cos r - 5z_s \right). \]  

(19)

As a result, (12) becomes

\[ \dot{x}_{1,s} = -x_{1,s} + g_{a,1} \]
\[ \dot{z}_s = -5z_s + (\cos(x_{2,s} + r) - 10) g_{a,2} \]  

(20)

Therefore, the closed-loop system (20) is input-to-state stable with respect to the input \( e_p \).
Controller Design

Combining the observer (13), (18) and (19) yields the final controller as

\[ u_p(s) = \frac{Q(s) e^{-Ts}}{1 - Q(s) e^{-Ts}} B(s) \hat{e}_p(s) \]

\[ u_s = -4\hat{x}_{1,s} + 8\hat{x}_{2,s} \]

\[ + \frac{1}{\cos(\hat{x}_{2,s} + r) - 10} \left( -2 (-\hat{x}_{1,s} + \hat{z}_s) - \cos(\hat{x}_{2,s} + r) \dot{r} + \dot{r} \cos r - 5\hat{z}_s \right) \]

\[ u = u_p + u_s \quad (21) \]

where \( \hat{e}_p = r - \hat{y}_p \) and \( \hat{z}_s = 6\hat{x}_{1,s} - 10\hat{x}_{2,s} + \sin(\hat{x}_{2,s} + r) - \sin r \). The tracking performance of the system (2) driven by the controller (21) is shown in Fig 7.3.
Controller Design

As shown, the output $y$ tracks $r$ with good tracking accuracy, meanwhile keeping all states bounded.

**Figure:** The amplitude response of the transfer function $G(s)$
In this chapter, the output feedback RC problem for a class of nonminimum-phase nonlinear systems was solved under the additive-state-decomposition-based tracking control framework. The key idea is to decompose the RC problem into two well-solved control problems by the additive state decomposition: an RC for a primary LTI system and a state feedback stabilizing control for a secondary nonlinear system.

Since the RC problem is only limited to an LTI system, existing RC methods can be applied directly and the resulting closed-loop system can be analyzed in the frequency domain.

On the other hand, the state feedback stabilization is only needed to consider for the ‘secondary’ stabilization problem so that the nonminimum-phase behavior is avoided.

Finally, one can combine the RC with the stabilizing controller to achieve the original control goal.
To show the differences among the three methods proposed so far, the design processes of the feedback linearization method, the adaptive-control-like method and the additive-state-decomposition-based method are different, shown in Fig. 7.8.

**Figure:** Three RC Design Approaches
Exercise

- It is noticed that the property of nonminimum-phase cannot be changed by output feedback. Why?
- Restate the step of the state additive decomposition in for the system in Example 7.1.
- Fill the step of the derivation of Theorem 7.2.
- Why is the closed-loop system (20) input-to-state stable with respect to the input $e_p$. 

All course source can be downloaded at http://rfly.buaa.edu.cn/publications.html. For More, please refer to the book:

Thank you!