Repetitive Control for Nonlinear Systems
Lesson 09 An Actuator-Focused Design Method

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Preface

What is actuator-focused viewpoint and how is it applied to repetitive control problems?

Outline

Motivation and Objective

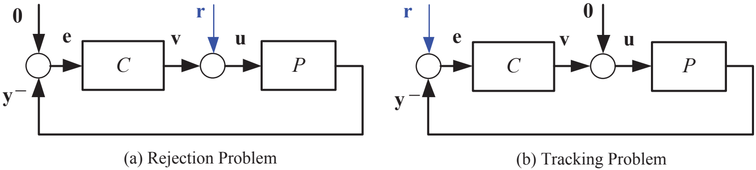


Figure: Tracking problem and rejection problem.

In Fig.1(a), the transfer function from r to e (corresponding to **r** and **e** respectively) is written as follows

$$e(s) = \frac{1}{1 - C(s)P(s)} r(s).$$

In Fig.1(b), the transfer function from r to e is written as follows

$$e(s) = \frac{1}{1 - C(s)P(s)} P(s) r(s).$$

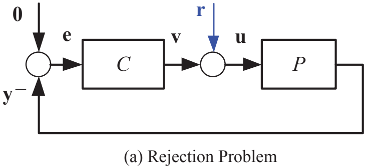


Figure: Tracking problem and rejection problem.

If all poles of $P(s)$ lie in the left s-plane, then the controller C only needs to ensure that all poles of $1/(1-C(s)P(s))$ lie in the left s-plane and one closed-loop zero of $1/(1-C(s)P(s))$ is 0 so that the closed-loop zero can cancel the unstable pole of $r(s)=a/s$, where a is constant. According to IMP, the controller C should contain an integral term $1/s$. Therefore, both tracking problem and rejection problem can be reduced to a stabilization problem.

Motivation and Objective

Objective. By taking these into account, a new viewpoint on IMP is proposed to support general periodic signal tracking of nonlinear systems

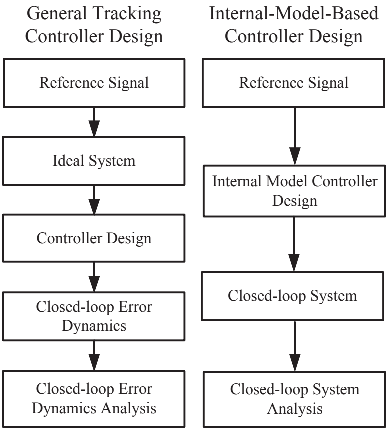


Figure: Comparison between two design ideas.

Actuator-Focused Viewpoint on IMP

(1) Step Signals

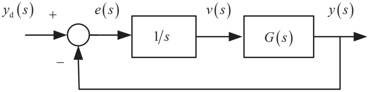


Figure: Step signal tracking.

Cancelation Viewpoint: As shown in Fig.4, the transfer function from the desired signal to the tracking error is written as follows

$$e(s) = \frac{1}{1+\frac{1}{s}G(s)} y_{\mathrm{d}}(s) = \frac{1}{s+G(s)}\left(s\frac{1}{s}\right) = \frac{1}{s+G(s)}.$$

Then it only requires to verify whether or not the roots of the equation $s+G(s)=0$ are all in the left s-plane. If all roots are in the left s-plane, then the tracking error tends to zero as $t \to \infty$. Therefore, the tracking problem has been reduced to a stability problem of the closed-loop system.

Three Examples

Actuator-Focused Viewpoint: This new viewpoint will give a new explanation on IMP without using transfer functions. Assume that the minimal realization of $y = G(s)v$ is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}v$$
$$y = \mathbf{c}^{\mathrm{T}}\mathbf{x} + dv$$

where $G(s) = \mathbf{c}^{\mathrm{T}}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d$. As shown in Fig.4, the resulting closed-loop system becomes

$$\underbrace{\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{v} \end{bmatrix}}_{\dot{\mathbf{z}}} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{b} \\ -\mathbf{c}^{\mathrm{T}} & -d \end{bmatrix}}_{\mathbf{A}_a}\underbrace{\begin{bmatrix} \mathbf{x} \\ v \end{bmatrix}}_{\mathbf{z}} + \underbrace{\begin{bmatrix} \mathbf{0} \\ y_{\mathrm{d}} \end{bmatrix}}_{\mathbf{w}}. \qquad (1)$$

The solution is

$$\mathbf{z}(t) = e^{\mathbf{A}_a t}\mathbf{z}(0) + \int_0^t e^{\mathbf{A}_a (t-s)}\mathbf{w}\mathrm{d}s$$

where $\mathbf{w}$ is constant. If the closed-loop system is stable, then the matrix $\mathbf{A}_a$ is stable, namely the real parts of eigenvalues of $\mathbf{A}_a$ are negative. As a result, $\mathbf{z}(t)$ will tend to a constant vector as $t \to \infty$. Consequently, $v(t)$ and $e(t) = y_{\mathrm{d}} - \mathbf{c}^{\mathrm{T}}\mathbf{x}$ will tend to constants v^* and e^* as $t \to \infty$, respectively. It can be claimed that $e^* = 0$. Otherwise, because of the integral term, one has

$$\dot{v}(t) = e(t) \tag{2}$$

$v(t)$ will tend to infinity. Therefore, to confirm that the tracking error tends to zero as $t \to \infty$, it is only required to verify whether or not the closed-loop system without external signals is exponentially stable. This implies that the tracking problem has been reduced to a stability problem.

(2)Sine Signals

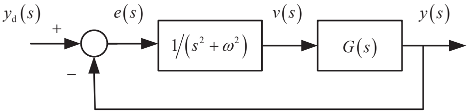


Figure: Sine signal tracking

Cancelation Viewpoint: As shown in Fig.6, the transfer function from the desired signal to the tracking error is written as follows

$$
\begin{aligned}
e(s) &= \frac{1}{1+\frac{1}{s^2+\omega^2}G(s)} y_d(s) \\
&= \frac{1}{s^2+\omega^2+G(s)} \left(\left(s^2+\omega^2\right) \frac{b_1 s + b_0}{s^2+\omega^2} \right) \\
&= \frac{b_1 s + b_0}{s^2+\omega^2+G(s)}
\end{aligned}
$$

(2)Sine Signals

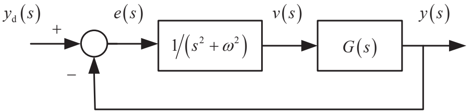


Figure: Sine signal tracking

where the Laplace transformation model of $a_0 \sin(\omega t + \varphi_0)$ is $\frac{b_1 s + b_0}{s^2+\omega^2}$. Then, it is only required to verify whether or not the roots of the equation $s^2 + \omega^2 + G(s) = 0$ are all in the left s-plane, namely whether or not the closed-loop system is stable. Therefore, the tracking problem has been reduced to a stability problem of the closed-loop system.

Actuator-Focused Viewpoint: Because of the term $\frac{1}{s^2+\omega^2}$, the relationship between $v(t)$ and $e(t)$ can be written as

$$e(t) = \ddot{v}(t) + \omega^2 v(t). \tag{3}$$

If the closed-loop system without external signals is exponentially stable, then, when the system is driven by an external signal in the form of $a_0 \sin(\omega t + \varphi_0)$, it is easy to see that $v(t)$ and $e(t)$ will tend to signals in the form of $a \sin(\omega t + \varphi)$, where a and φ are constants. Consequently,

$$e(t) \to (a \sin(\omega t + \varphi))'' + \omega^2 (a \sin(\omega t + \varphi)) \equiv 0$$

as $t \to \infty$ by (3) no matter what a and φ are. Therefore, to confirm that the tracking error tends to zero as $t \to \infty$, it only requires verifying whether or not the closed-loop system without external signals is exponentially stable. This implies that the tracking problem has been reduced to a stability problem.

(3)General T-Periodic Signal

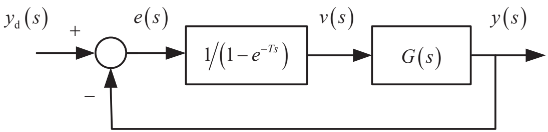


Figure: Periodic signal tracking of an RC system.

Cancelation Viewpoint: Similarly, as shown in Fig.8, the transfer function from the desired signal to the error is written as follows

$$\begin{aligned} e(s) &= \frac{1}{1+\frac{1}{1-e^{-sT}}G(s)} y_d(s) \\ &= \frac{1}{1-e^{-sT}+G(s)} \left(\left(1-e^{-sT}\right) \frac{1}{1-e^{-sT}} \right) \\ &= \frac{1}{1-e^{-sT}+G(s)}. \end{aligned}$$

(3)General T-Periodic Signal

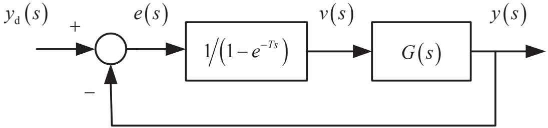


Figure: Periodic signal tracking of an RC system.

Then, it is only required to verify whether or not the roots of the equation $1 - e^{-sT} + G(s) = 0$ are all in the left s-plane. Therefore, the tracking problem has been reduced to a stability problem of the closed-loop system.

Actuator-Focused Viewpoint: Because of the term $1\left/\left(1-e^{-sT}\right)\right.$, the relationship between $v\left(t\right)$ and $e\left(t\right)$ can be written as

$$e\left(t\right)=v\left(t\right)-v\left(t-T\right). \tag{4}$$

If the closed-loop system without external signals is exponentially stable, then, by the solution of $v\left(t\right)$ and $e\left(t\right)$, it can be proven that $v\left(t\right)$ and $e\left(t\right)$ will both tend to T-periodic signals when the system is driven by a T-periodic signal. Consequently, it can be concluded that $e\left(t\right)\to 0$ as $t\to\infty$ by (4). Therefore, to examine the tracking error tending to zero as $t\to\infty$, it only requires verifying whether or not the closed-loop system without external signals is exponentially stable. This implies that the tracking problem has been reduced to a stability problem.

Filtered Repetitive Control Systems Subject to T-Periodic Signals

The model $Q(s)/(1 - Q(s)e^{-sT})$ replaces $1/(1 - e^{-sT})$ resulting in the closed-loop system shown in Fig.9. Furthermore, if $Q(s) = 1/(1 + \epsilon s)$, then the relationship between $v(t)$ and $e(t)$ is

$$e(t) = v(t) - v(t - T) + \epsilon \dot{v}(t). \tag{5}$$

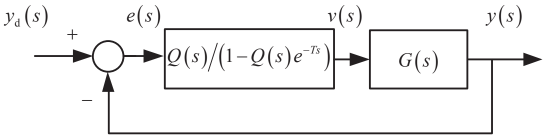


Figure: Periodic signal tracking of an FRC system.

Filtered Repetitive Control Systems Subject to T-Periodic Signals

- If the closed-loop system without external signals is exponentially stable, then, when the system is driven by a periodic signal, it is easy to see that $v(t)$ and $e(t)$ will both tend to periodic signals as $t \to \infty$.
- Because of the relationship (5), it can be concluded that $e(t) - \epsilon \dot{v}(t) \to 0$. This implies that the tracking error can be adjusted by the filter $Q(s)$ or say ϵ. Moreover, if $\dot{v}(t)$ is bounded in t uniformly with respect to (w.r.t) ϵ as $\epsilon \to 0$, then $\lim_{t\to\infty,\epsilon\to 0} e(t,\epsilon) = 0$.
- On the other hand, increasing ϵ can improve the stability of the closed-loop system.
- Therefore, a satisfactory tradeoff between stability and tracking performance can be achieved by using the FRC.

Actuator-Focused RC Design Method

Linear Periodic System

Consider the following linear periodic system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{d}(t)\\ \mathbf{y}(t) &= \mathbf{C}^{\mathrm{T}}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)\end{aligned} \tag{6}$$

where matrices $\mathbf{A}(t+T) = \mathbf{A}(t) \in \mathbb{R}^{n\times n}$, $\mathbf{B}(t+T) = \mathbf{B}(t) \in \mathbb{R}^{n\times m}$, $\mathbf{C}(t+T) = \mathbf{C}(t) \in \mathbb{R}^{n\times m}$, and $\mathbf{D}(t+T) = \mathbf{D}(t) \in \mathbb{R}^{m\times m}$ are bounded; $\mathbf{x}(t) \in \mathbb{R}^n$ is the system state, $\mathbf{u}(t) \in \mathbb{R}^m$ is the control input, $\mathbf{d} \in \mathcal{C}_T^0([0,\infty), \mathbb{R}^m)$ is a T-periodic disturbance. The objective of the control input $\mathbf{u}$ is to make $\mathbf{y}(t)$ track a T-periodic desired signal $\mathbf{y}_{\mathrm{d}} \in \mathcal{C}_T^0([0,\infty); \mathbb{R}^m)$.

Linear Periodic System

For the system (6), similar to equation (5), an FRC is taken in the form as

$$\begin{aligned} \mathbf{A}_{\epsilon}\dot{\mathbf{v}}(t) &= -\mathbf{v}(t) + (\mathbf{I}_m - \alpha\mathbf{A}_{\epsilon})\mathbf{v}(t-T) + \mathbf{L}_1(t)\mathbf{e}(t) \\ \mathbf{u}(t) &= \mathbf{L}_2(t)\mathbf{x}(t) + \mathbf{v}(t) \end{aligned} \tag{7}$$

where $\mathbf{e} \triangleq \mathbf{y}_{\mathrm{d}} - \mathbf{y}$, $\mathbf{A}_{\epsilon} \in \mathbb{R}^{m\times m}$ is a positive definite matrix, $\alpha > 0$, $\mathbf{L}_1(t+T) = \mathbf{L}_1(t)$ is nonsingular and $\mathbf{L}_2(t+T) = \mathbf{L}_2(t)$. Moreover, $\mathbf{L}_1(t)$ and $\mathbf{L}_2(t)$ are bounded. Then

$$\mathbf{y}(t) = \left(\mathbf{C}^{\mathrm{T}}(t) + \mathbf{D}(t)\mathbf{L}_2(t)\right)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{v}(t).$$

Next, by combining the system (6) and FRC (7), the resulting closed-loop system is written as follows

$$\mathbf{E}\dot{\mathbf{z}}(t) = \mathbf{A}_a(t)\mathbf{z}(t) + \mathbf{A}_{a,-T}\mathbf{z}(t-T) + \mathbf{B}_a(t)\mathbf{w}(t). \tag{8}$$

Linear Periodic System

Consider a general perturbed time-delay system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}_t, \mathbf{w}), t \geq t_0 \tag{9}$$

with $\mathbf{x}_{t_0}(s) = \phi(s), s \in [-\tau, 0], \tau \in \mathbb{R}_+$, where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{w}(t) \in \mathbb{R}^m$ is a piecewise continuous and bounded perturbation. The function $\mathbf{f} : [t_0, \infty) \times \mathcal{C}([-\tau, 0], \mathbb{R}^n) \times \mathbb{R}^m \to \mathbb{R}^n$ is supposed to be continuous and takes bounded sets into bounded sets. Here, let initial time $t_0 = 0$ for simplicity.

Lemma 9.1

For (9), suppose (i) $\mathbf{f}(t, \mathbf{x}_t, \mathbf{w}(t)) = \mathbf{f}(t + T, \mathbf{x}_t, \mathbf{w}(t + T))$, (ii) $\mathbf{f}(t, \mathbf{x}_t, \mathbf{w})$ satisfies a local Lipschitz condition in $\mathbf{x}_t$, (iii) $\mathbf{x}(t + T)$ is a solution of (9) whenever $\mathbf{x}(t)$ is a solution of (9). If solutions of (9) are uniformly bounded and uniformly ultimately bounded, then (9) has a T-periodic solution.

Linear Periodic System

Lemma 9.2

Suppose that the solution $\mathbf{z}(t) = \mathbf{0}$ of the differential equation

$$\mathbf{E}\dot{\mathbf{z}}(t) = \mathbf{A}_a(t)\,\mathbf{z}(t) + \mathbf{A}_{a,-T}\mathbf{z}(t-T) \qquad (10)$$

is globally exponentially stable. Then the resulting closed-loop system in (8) has a unique globally exponentially stable T-periodic solution $\mathbf{z}^*$.

Theorem 9.2

Suppose that (10) is globally exponentially stable. Then, the resulting closed-loop system in (8) has a T-periodic solution $\mathbf{z}^* = [\mathbf{v}^{*\mathrm{T}}\ \mathbf{x}^{*\mathrm{T}}]^{\mathrm{T}}$. Furthermore,

$$\|\mathbf{e}\|_a \leq \sup_{t\in[0,T]} \left\|\mathbf{L}_1^{-1}(t)\,\mathbf{A}_\epsilon\right\| \left(\|\dot{\mathbf{v}}\|_a + \alpha\,\|\mathbf{v}\|_a\right).$$

If $\mathbf{z}(t) = \mathbf{0}$ in (10) is globally exponentially stable uniformly w.r.t $\mathbf{A}_\epsilon$ as $\|\mathbf{A}_\epsilon\| \to 0$, then $\lim_{t\to\infty, \|\mathbf{A}_\epsilon\|\to 0} \|\mathbf{e}(t,\mathbf{A}_\epsilon)\|_a = 0$.

Proof: By *Lemma 9.2*, the resulting closed-loop system in (8) has a unique globally exponentially stable T-periodic solution $\mathbf{z}^*$. By using (7), it follows that

$$\mathbf{L}_1(t)\,\mathbf{e}(t) = \mathbf{A}_\epsilon \dot{\mathbf{v}}(t) + \mathbf{v}(t) - (1 - \alpha \mathbf{A}_\epsilon)\,\mathbf{v}(t - T).$$

Taking $\|\cdot\|_a$ on both sides of the equation above yields

$$\begin{aligned}\|\mathbf{e}\|_a &= \limsup_{t\to\infty} \left\| \mathbf{L}_1^{-1}(t)\,\mathbf{A}_\epsilon(\dot{\mathbf{v}}(t) + \alpha \mathbf{v}(t - T)) + \mathbf{L}_1^{-1}(t)(\mathbf{v}(t) - \mathbf{v}(t - T)) \right\| \\ &\le \sup_{t\in[0,T]} \left\| \mathbf{L}_1^{-1}(t)\,\mathbf{A}_\epsilon \right\| \left(\|\dot{\mathbf{v}}\|_a + \alpha \|\mathbf{v}\|_a \right).\end{aligned}$$

If (10) is globally exponentially stable uniformly w.r.t $\mathbf{A}_\epsilon$ as $\|\mathbf{A}_\epsilon\| \to 0$, then $\|\dot{\mathbf{v}}\|_a + \alpha \|\mathbf{v}\|_a$ is bounded uniformly w.r.t $\mathbf{A}_\epsilon$ as $\|\mathbf{A}_\epsilon\| \to 0$. Consequently, $\|\mathbf{A}_\epsilon\| \left(\|\dot{\mathbf{v}}\|_a + \alpha \|\mathbf{v}\|_a \right) \to 0$ as $\|\mathbf{A}_\epsilon\| \to 0$. This implies that $\|\mathbf{e}(t, \mathbf{A}_\epsilon)\|_a \to 0$ as $\|\mathbf{A}_\epsilon\| \to 0$. $\square$

Linear Periodic System

Theorem 9.3

If there exist matrices $0 < \mathbf{P} = \mathbf{P}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, $0 < \mathbf{Q} = \mathbf{Q}^{\mathrm{T}} \in \mathbb{R}^{m\times m}$, $\lambda_1 \in \mathbb{R}_+$ such that

$$0 < \mathbf{PE} + \mathbf{E}^{\mathrm{T}}\mathbf{P} \tag{11}$$

$$\begin{bmatrix} \mathbf{PA}_a(t) + \mathbf{A}_a^{\mathrm{T}}(t)\mathbf{P} + \mathbf{Q} & \mathbf{PA}_{a,-T} \\ \mathbf{A}_{a,-T}^{\mathrm{T}}\mathbf{P} & -\mathbf{Q} \end{bmatrix} \leq -\lambda_1 \mathbf{I}_{n+m} \tag{12}$$

then $\mathbf{z}(t) = \mathbf{0}$ in (10) is globally exponentially stable when $0 < \mathbf{A}_\epsilon$. Furthermore, if there exists $\lambda_2 \in \mathbb{R}_+$ such that

$$\sup_{t\in[0,T]} \left\| \left(\mathbf{I}_m + \mathbf{L}(t)\mathbf{D}(t)\right)^{-1} \right\| < 1, \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \lambda_2 \mathbf{I}_{n+m} \end{bmatrix} \leq \mathbf{PE} + \mathbf{E}^{\mathrm{T}}\mathbf{P}$$

then $\mathbf{z}(t) = \mathbf{0}$ in (10) is globally exponentially stable when $\mathbf{A}_\epsilon = \mathbf{0}$.

General Nonlinear System

In the following, let us consider a general perturbed nonlinear system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}\left(t, \mathbf{x}, \mathbf{u}, \mathbf{d}\right) \\ \mathbf{y} &= \mathbf{g}\left(\mathbf{x}, \mathbf{u}\right) \end{aligned} \tag{13}$$

where $\mathbf{f} : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$, $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, and $\mathbf{f}\left(t, \mathbf{x}, \mathbf{u}, \mathbf{d}\left(t\right)\right) = \mathbf{f}\left(t + T, \mathbf{x}, \mathbf{u}, \mathbf{d}\left(t + T\right)\right)$; $\mathbf{x}\left(t\right) \in \mathbb{R}^n$ is the system state, $\mathbf{u}\left(t\right) \in \mathbb{R}^m$ is the control input, $\mathbf{d} \in \mathcal{C}_T^0\left([0, \infty), \mathbb{R}^m\right)$ is the T-periodic disturbance. The objective of the control input $\mathbf{u}$ is to make $\mathbf{y}\left(t\right)$ track T-periodic desired signal $\mathbf{y}_\mathrm{d} \in \mathcal{C}_T^0\left([0, \infty); \mathbb{R}^m\right)$.

General Nonlinear System

For the system (13), similar to (5), an FRC is taken in the form as

$$\begin{aligned} \mathbf{A}_{\epsilon}\dot{\mathbf{v}}(t) &= -\mathbf{v}(t) + (1-\alpha\mathbf{A}_{\epsilon})\mathbf{v}(t-T) + \mathbf{h}(t,\mathbf{e}) \\ \mathbf{u}(t) &= \mathbf{u}_{\mathrm{st}}(\mathbf{x}(t)) + \mathbf{v}(t) \end{aligned} \qquad (14)$$

where $\mathbf{e} \triangleq \mathbf{y}_{\mathrm{d}} - \mathbf{y}$, $\mathbf{A}_{\epsilon} \in \mathbb{R}^{m\times m}$ is a positive definite matrix, $\alpha > 0$, $\mathbf{h} : \mathbb{R}^{m} \times \mathbb{R}^{m} \to \mathbb{R}^{m}$ is a continuous function, and $\mathbf{u}_{\mathrm{st}} : \mathbb{R}^{n} \to \mathbb{R}^{m}$ is a state feedback law employed to stabilize the state of the considered plant (13). The functions $\mathbf{h}(\cdot)$ and $\mathbf{u}_{\mathrm{st}}(\cdot)$ are both locally Lipschitz. On the other hand, the continuous function $\mathbf{v}$ represents a feedforward input which will drive the output $\mathbf{y}$ of (13) to track the given desired trajectory $\mathbf{y}_{\mathrm{d}}$.

Next, the resulting closed-loop system is written as follows

$$\mathbf{E}\dot{\mathbf{z}} = \mathbf{f}_a\left(t, \mathbf{z}_t, \mathbf{w}\right) \tag{15}$$

where

$$\begin{aligned}
\mathbf{z} &= \left[\begin{array}{cc} \mathbf{v}^{\mathrm{T}} & \mathbf{x}^{\mathrm{T}} \end{array}\right]^{\mathrm{T}}, \mathbf{w} = \left[\begin{array}{cc} \mathbf{y}_{\mathrm{d}}^{\mathrm{T}} & \mathbf{d}^{\mathrm{T}} \end{array}\right]^{\mathrm{T}} \\
\mathbf{E} &= \mathrm{diag}\left(\mathbf{A}_{\epsilon}, \mathbf{I}_n\right), \mathbf{y} = \mathbf{g}\left(\mathbf{x}, \mathbf{u}_{\mathrm{st}}\left(\mathbf{x}\right) + \mathbf{v}\right) \\
\mathbf{f}_a\left(t, \mathbf{z}_t, \mathbf{w}\right) &= \left[\begin{array}{c} -\mathbf{v} + \left(1 - \alpha\mathbf{A}_{\epsilon}\right)\mathbf{v}\left(t - T\right) - \mathbf{h}\left(t, \mathbf{e}\right) \\ \mathbf{f}\left(t, \mathbf{x}, \mathbf{u}_{\mathrm{st}}\left(\mathbf{x}\right) + \mathbf{v}, \mathbf{d}\right) \end{array}\right].
\end{aligned}$$

General Nonlinear System

Theorem 9.4

Suppose (i) the solutions of the resulting closed-loop system in (15) are uniformly bounded and uniformly ultimately bounded; (ii) $\mathbf{h}(t, \mathbf{e}) \to \mathbf{0}$ implies $\mathbf{e} \to \mathbf{0}$. Then the resulting closed-loop system in (15) has a T-periodic solution $\mathbf{z}^* = [\mathbf{v}^{*\mathrm{T}}\ \mathbf{x}^{*\mathrm{T}}]^{\mathrm{T}}$. Furthermore, if

$$\mathbf{E}\dot{\mathbf{z}}_{\mathrm{e}} = \mathbf{f}_a(t, \mathbf{z}_t^* + \mathbf{z}_{\mathrm{e}t}, \mathbf{w}) - \mathbf{f}_a(t, \mathbf{z}_t^*, \mathbf{w}) \tag{16}$$

is locally (globally) exponentially stable, then the T-periodic solution $\mathbf{z}^*$ is locally (globally) exponentially stable and

$$\|\mathbf{h}(t, \mathbf{e})\|_a \leq \|\mathbf{A}_\epsilon\| \left(\|\dot{\mathbf{v}}\|_a + \alpha \|\mathbf{v}\|_a\right)$$

holds locally (globally), where $\mathbf{z}_{\mathrm{e}} \triangleq \mathbf{z} - \mathbf{z}^*$. Furthermore, if $\|\dot{\mathbf{v}}(t, \mathbf{A}_\epsilon)\|_a$ and $\|\mathbf{v}(t, \mathbf{A}_\epsilon)\|_a$ are bounded in t uniformly w.r.t $\mathbf{A}_\epsilon$ as $\|\mathbf{A}_\epsilon\| \to 0$, then $\lim_{t\to\infty, \|\mathbf{A}_\epsilon\|\to 0} \|\mathbf{e}(t, \mathbf{A}_\epsilon)\|_a = 0$ locally (globally).

General Nonlinear System

- The major idea of the actuator-focused RC design is to make $\mathbf{h}(t, \mathbf{e})$ as the input of the internal model, i.e. $\mathbf{A}_{\epsilon}\dot{\mathbf{v}}(t) = -\mathbf{v}(t) + (1 - \alpha\mathbf{A}_{\epsilon})\,\mathbf{v}(t - T) + \mathbf{h}(t, \mathbf{e})$. If the closed-loop system tends to equilibrium, then the tracking error can be analyzed according to the RC itself. This is based on the actuator-focused viewpoint.
- The major advantage of the proposed actuator-focused RC design is to avoid the derivation of error dynamics. This facilitates the tracking controller design.
- The designed controller is applied not only to the rejection problem but also to the tracking problem. Through incorporating the internal model into the closed-loop system, it is only necessary to ensure that the latter is uniformly bounded and uniformly ultimately bounded.

Numerical Examples

(1) A Linear Periodic System
Consider the following linear periodic system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 - 0.5\sin t & -2 - \cos t \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0.5\sin t \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \sin(t+1) \end{bmatrix}$$
$$y(t) = \begin{bmatrix} 1 & \cos t \end{bmatrix} \mathbf{x}(t) + u(t). \quad (17)$$

The objective is to design u to drive the signal $y(t) - y_{\mathrm{d}}(t) \to 0$, where $y_{\mathrm{d}}(t) = \sin t$ for simplicity. For the system above, according to FRC (7), design

$$\begin{aligned} v(t) &= v(t-T) + L_1 (y_{\mathrm{d}}(t) - y(t)) \\ u(t) &= v(t), v(s) = 0, s \in [-T, 0] \end{aligned} \quad (18)$$

where $L_1 = 6$.

Numerical Examples

By the actuator-focused viewpoint, the control form (18) is to establish an input-output relation as follows

$$y_{\mathrm{d}}(t) - y(t) = \frac{1}{L_1}\left(v(t) - v(t - T)\right).$$

Since v approaches a T-periodic signal, it can be concluded that $y_{\mathrm{d}}(t) - y(t) \to 0$ as $t \to \infty$.

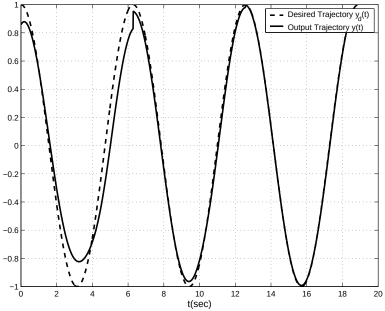


Figure: Linear periodic system tracking

A Minimum-Phase Nonlinear System

The dynamics of an m-degree-of-freedom manipulator are described by the following differential equation

$$\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \mathbf{u} \tag{19}$$

where $\mathbf{q} \in \mathbb{R}^m$ denotes the vector of generalized displacements in robot co-ordinates, $\mathbf{u} \in \mathbb{R}^m$ denotes the vector of generalized control input forces in robot coordinates; $\mathbf{D}(\mathbf{q}) \in \mathbb{R}^{m\times m}$ is the manipulator inertial matrix, $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^{m\times m}$ is the vector of centripetal and Coriolis torques and $\mathbf{G}(\mathbf{q}) \in \mathbb{R}^m$ is the vector of gravitational torques. It is assumed that both $\mathbf{q}$ and $\dot{\mathbf{q}}$ are available from measurements. Because of no internal dynamics, the system (19) is a minimum-phase nonlinear system. Two common assumptions in the following are often made on the system (19).

A Minimum-Phase Nonlinear System

(A1) The inertial matrix $\mathbf{D}(\mathbf{q})$ is symmetric, uniformly positive definite and bounded, i.e.,

$$0 < \underline{\lambda}_D \mathbf{I}_m \leq \mathbf{D}(\mathbf{q}) \leq \bar{\lambda}_D \mathbf{I}_m, \forall \mathbf{q} \in \mathbb{R}^m \tag{20}$$

where $\underline{\lambda}_D, \bar{\lambda}_D \in \mathbb{R}_+$.
(A2) The matrix $\dot{\mathbf{D}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is skew-symmetric, hence

$$\mathbf{x}^{\mathrm{T}}\left(\dot{\mathbf{D}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\right)\mathbf{x} = 0, \forall \mathbf{x} \in \mathbb{R}^m.$$

For a given desired trajectory $\mathbf{q}_{\mathrm{d}} \in \mathcal{C}_{PT}^2([0, \infty), \mathbb{R}^m)$, the controller $\mathbf{u}$ is designed to make $\mathbf{q}$ track $\mathbf{q}_{\mathrm{d}}$. Define a new state $\mathbf{x}$ as follows

$$\mathbf{x} = \dot{\mathbf{q}} + \mu \mathbf{q}$$

where $\mu \in \mathbb{R}_+$.

A Minimum-Phase Nonlinear System

According to (14), a control law $\mathbf{u}$ is taken in the form as

$$\begin{aligned}\epsilon\dot{\mathbf{v}}(t) &= -\mathbf{v}(t) + (1-\alpha\epsilon)\mathbf{v}(t-T) + k((\dot{\mathbf{q}}_{\mathrm{d}} + \mu\mathbf{q}_{\mathrm{d}}) - \mathbf{x})(t) \\ \mathbf{u}(t) &= \mathbf{v}(t) - \mathbf{M}\mathbf{x}(t) + \mathbf{G}(\mathbf{q}(t)) - \mu\mathbf{D}(\mathbf{q}(t))\dot{\mathbf{q}}(t) - \mu\mathbf{C}(\mathbf{q}(t), \dot{\mathbf{q}}(t))\mathbf{q}(t)\end{aligned} \tag{21}$$

where $\mathbf{v}(s) = \mathbf{0}, s \in [-T, 0], 0 < \mathbf{M} = \mathbf{M}^{\mathrm{T}} \in \mathbb{R}^{m\times m}$ is positive definite matrix and $k \in \mathbb{R}_{+}$. Substituting the controller (21) into (19) results in

$$\begin{aligned}\epsilon\dot{\mathbf{v}}(t) &= -\mathbf{v}(t) + (1-\alpha\epsilon)\mathbf{v}(t-T) + k((\dot{\mathbf{q}}_{\mathrm{d}} + \mu\mathbf{q}_{\mathrm{d}}) - \mathbf{x})(t) \\ \dot{\mathbf{x}}(t) &= -\mathbf{D}^{-1}(\mathbf{q}(t))(\mathbf{C}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) + \mathbf{M}(t))\mathbf{x}(t) + \mathbf{D}^{-1}(\mathbf{q}(t))\mathbf{v}(t).\end{aligned} \tag{22}$$

A Minimum-Phase Nonlinear System

The closed-loop system (22) can be rewritten in the form of (15) with

$$\mathbf{z} = \begin{bmatrix} \mathbf{v}^{\mathrm{T}} & \mathbf{x}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \mathbf{E} = \operatorname{diag}\left(\epsilon \mathbf{I}_m, \mathbf{I}_n\right)$$
$$\mathbf{w} = \begin{bmatrix} \mathbf{x}_{\mathrm{d}}^{\mathrm{T}} & \mathbf{0} \end{bmatrix}^{\mathrm{T}}, \mathbf{x}_{\mathrm{d}} = \dot{\mathbf{q}}_{\mathrm{d}} + \mu \mathbf{q}_{\mathrm{d}} \tag{23}$$
$$\mathbf{f}_a\left(t, \mathbf{z}_t, \mathbf{w}\right) = \mathbf{E}^{-1} \begin{bmatrix} -\mathbf{v}\left(t\right) + \left(1 - \alpha\epsilon\right) \mathbf{v}\left(t - T\right) + k\left(\left(\dot{\mathbf{q}}_{\mathrm{d}} + \mu \mathbf{q}_{\mathrm{d}}\right) - \mathbf{x}\right)\left(t\right) \\ -\mathbf{D}^{-1}\left(\mathbf{q}\left(t\right)\right)\left(\mathbf{C}\left(\mathbf{q}\left(t\right), \dot{\mathbf{q}}\left(t\right)\right) + \mathbf{M}\right)\mathbf{x}\left(t\right) + \mathbf{D}^{-1}\left(\mathbf{q}\left(t\right)\right)\mathbf{v}\left(t\right) \end{bmatrix}.$$

Theorem 5

Suppose (i) Assumptions (A1)-(A2) hold, (ii) $0 < \alpha\epsilon < 1, \epsilon, \alpha, k \in \mathbb{R}_+$. Then the solutions of the closed-loop system (15) with (23) are uniformly bounded and uniformly ultimately bounded.

A Nonminimum-Phase Nonlinear System

Consider the following nonlinear system

$$\begin{aligned} \dot{\eta} &= \sin\eta + \xi + d_\eta \\ \dot{\xi} &= u + d_\xi \\ y &= \xi \end{aligned} \tag{24}$$

where $\eta(t), \xi(t), y(t) \in \mathbb{R}$, $d_\eta, d_\xi \in \mathcal{C}_T^0([0,\infty), \mathbb{R}^m)$ are T-periodic disturbances. Since zero dynamics $\dot{\eta} = \sin\eta$ is unstable, the system (24) is a *nonminimum-phase* nonlinear system. The control is required not only to cause y to track y_d, but also to make the internal dynamics bounded.

A Nonminimum-Phase Nonlinear System

Compared with the existing design, the proposed design method will simplify the controller design. According to (14), a control law u is taken in the form as

$$\begin{aligned} \epsilon \dot{v}(t) &= -v(t) + (1-\alpha\epsilon)\, v(t-T) + k(y_{\mathrm{d}} - y)(t) \\ u(t) &= -(q_1 + \cos\eta)(-q_1\eta + z)(t) - \rho z(t) - q_2\eta(t) + v(t) \end{aligned} \tag{25}$$

where $v(s) = 0, s \in [-T, 0]$, $v, \alpha, \epsilon, k, q_1, q_2, \rho \in \mathbb{R}$ and $z = \xi + q_1\eta + \sin\eta$. Substituting the controller (25) into (24) results in

$$\begin{aligned} \epsilon \dot{v}(t) &= -v(t) + (1-\alpha\epsilon)\, v(t-T) - k(z - q_1\eta - \sin\eta)(t) + k y_{\mathrm{d}}(t) \\ \dot{\eta}(t) &= -q_1\eta(t) + z(t) + d_\eta(t) \\ \dot{z}(t) &= -kz(t) - q_2\eta(t) + v(t) + d_\xi(t) + d_\eta(t)(q_1 + \cos\eta)(t). \end{aligned} \tag{26}$$

A Nonminimum-Phase Nonlinear System

From the simulation, v approaches a T-periodic solution, then

$$(y_{\mathrm{d}} - y)(t) - \frac{\epsilon}{k}(\dot{v} + \alpha v)(t) \to 0.$$

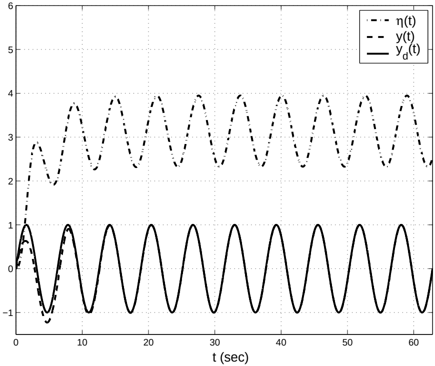


Figure: Periodic signal tracking of an FRC system.

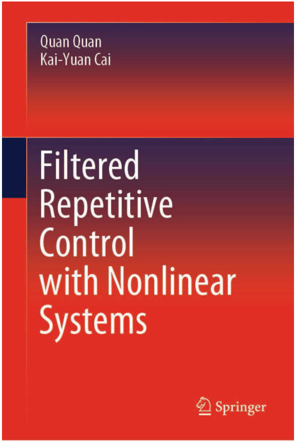
Summary

- A new viewpoint, namely actuator-focused design, on IMP is proposed. It can be used to explain how internal models work in the time domain.
- Guided by the actuator-focused viewpoint, the actuator-focused RC design method is further proposed for periodic signal tracking.
- In the controller design, the stability of the closed-loop system needs to be considered rather than that of the error dynamics.
- In order to demonstrate its effectiveness, the proposed design method is applied to RC problems for a linear periodic system (time-varying), a minimum-phase nonlinear system and a nonminimum-phase nonlinear system.

Exercise

- Fill the steps of the derivation of *Lemma 9.2.*
- Fill the steps of the derivation of *Theorem 9.2*
- Fill the steps of the derivation of *Theorem 9.3*
- Fill the steps of the derivation of *Theorem 9.4*.
- Fill the steps of the derivation of *Theorem 9.5*

All course source can be downloaded at http://rfly.buaa.edu.cn/publications.html. For More, please refer to the book:



Thank you!